ALGEBRAIC INTEGRABILITY CONDITIONS
FOR KILLING TENSORS
ON CONSTANT SECTIONAL CURVATURE MANIFOLDS

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Abstract. We use an isomorphism between the space of valence two Killing tensors on an n-dimensional constant sectional curvature manifold and the irreducible $\mathrm{GL}(n+1)$-representation space of algebraic curvature tensors [MMS04] in order to translate the Nijenhuis integrability conditions for a Killing tensor into purely algebraic integrability conditions for the corresponding algebraic curvature tensor, resulting in two simple algebraic equations of degree two and three. As a first application of this we construct a new family of integrable Killing tensors.

1. Introduction

Besides the Euler-Lagrange formalism and the Hamilton formalism, the Hamilton-Jacobi equation is one of the three fundamental reformulations of classical Newtonian mechanics with wide applications in physics as well as mathematics, ranging from classical mechanics over optics and semi-classical quantum mechanics to Riemannian geometry. In many cases this first-order non-linear partial differential equation can be solved by a separation of variables after choosing appropriate coordinates. It is therefore a classical problem in Riemannian geometry to classify such coordinates [Stä97, LC04, Eis34, KJ80]. The Hamilton-Jacobi equation separates in a given system of orthogonal coordinates if and only if there exists an integrable valence two Killing tensor field with simple eigenvalues whose eigenvectors are tangent to the coordinate lines and such that the potential satisfies a certain compatibility condition involving this Killing tensor [Ben93]. Integrable Killing tensors are thus an important tool in the study of the seprarability of the Hamilton-Jacobi equation. The present work focusses on the integrability condition for Killing tensors.

Killing tensors form a linear space which is invariant under the pullback action of the manifold's isometry group. In other words they constitute a representation space of the isometry group. McLenaghan, Milson and Smirnov identified this representation in the case of constant sectional curvature manifolds as a certain irreducible representation of the general linear group [MMS04]. More precisely, they used the isometric embeddings of the standard models of constant sectional curvature manifolds as hypersurfaces M in a Euclidean vector space (V, g) in order to write Killing tensors as restrictions of homogeneous polynomials on V, where the coefficients obey certain symmetry relations. This yields in particular an explicit natural isomorphism between the space of valence two Killing tensors on M and

the irreducible $\mathrm{GL}(V)$-representation space of algebraic curvature tensors on the ambient space V. Algebraic curvature tensors are valence four tensors subject to the symmetries of a Riemannian curvature tensor. Furthermore, this isomorphism is equivariant with respect to the action of the isometry group as a subgroup of $\mathrm{GL}(V)$.

This is the starting point for the present work: If the Killing tensor fields on a constant sectional curvature manifold correspond bijectively to algebraic curvature tensors, i. e. simple algebraic objects, then their integrability must be expressible as a purely algebraic condition on algebraic curvature tensors. This idea leads finally – after some tensor calculus based on results from the theory of representations of the symmetric and general linear groups – to the following simple algebraic integrability conditions:

Main Theorem. *A Killing tensor on a constant sectional curvature manifold M is integrable if and only if the associated algebraic curvature tensor R on V satisfies the following two conditions:*

$$\boxed{\begin{smallmatrix}a_2\\b_2\\c_2\\d_2\end{smallmatrix}}\;\bar{g}_{ij}R^i{}_{b_1\underline{a_2}\underline{b_2}}R^j{}_{d_1\underline{c_2}\underline{d_2}} = 0 \tag{1.1a}$$

$$\boxed{\begin{smallmatrix}a_2\\b_2\\c_2\\d_2\end{smallmatrix}}\;\boxed{\begin{smallmatrix}a_1&b_1&c_1&d_1\end{smallmatrix}}\;\bar{g}_{ij}\bar{g}_{kl}R^i{}_{b_1\underline{a_2}\underline{b_2}}R^j{}_{a_1}{}^k{}_{c_1}R^l{}_{d_1\underline{c_2}\underline{d_2}} = 0\,, \tag{1.1b}$$

where the operators on the left hand side are the Young symmetrisers for complete antisymmetrisation in the (underlined) indices a_2, b_2, c_2, d_2 respectively complete symmetrisation in the indices a_1, b_1, c_1, d_1. The tensor $\bar{g}$ denotes the inner product g on V in case M is not flat. Otherwise, i. e. if $M \subset V$ is a hyperplane, $\bar{g}$ is the (degenerated) pullback of g via the orthogonal projection $V \to M$.

This aproach to integrability of Killing tensor fields on constant sectional curvature manifolds has a certain number of advantages. The first and certainly the most important is, that we replace the Nijenhuis integrability conditions – a complicated non-linear system of partial differential equations for a tensor field on a manifold – by two simple algebraic equations for a tensor on a vector space. On the one hand this simplifies a numerical treatment considerably. Note that integrability can be checked by a simple evaluation of polynomials of degree two and three. On the other hand this opens the way for algebraic methods. Our formulation for example allowed us to show that the third of the Nijenhuis integrability conditions is redundant for Killing tensors on constant sectional curvature manifolds. In the special case of Euclidean 3-space, this result was already mentioned in a footnote of [HMS05], stating "Steve Czapor (private communication) has simplified the situation considerably. Using Gröbner basis theory, he has shown that (4.4a) and (4.4b) imply (4.4c), for any Killing tensor $\mathbf{K} \in \mathcal{K}^2(\mathbb{E}^3)$."[1]

Moreover, we can exhibit a family of integrable Killing tensors which arises naturally from our algebraic description and extends a known family [IMM00, BM03] which is based on the work of Benenti [Ben92]:

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[1] Equations (4.4) therein are the Nijenhuis integrability conditions, c. f. (2.3) here.

Main Corollary. *There exists a family of integrable Killing tensors on a non-zero constant sectional curvature manifold, given by the algebraic curvature tensors*

$$R = \lambda_2 h \owedge h + \lambda_1 h \owedge g + \lambda_0 g \owedge g \qquad h \in \mathrm{Sym}^2 V \qquad \lambda_0, \lambda_1, \lambda_2 \in \mathbb{R}. \qquad (1.2)$$

Here $\mathrm{Sym}^2 V$ is the space of symmetric 2-tensors[2] and $\owedge$ denotes the Kulkarni-Nomizu product. The corresponding Killing tensors read in local coordinates

$$\begin{aligned} K_{\alpha\beta} = {} & 2\lambda_2 (h_{a_1 a_2} h_{b_1 b_2} - h_{a_1 b_2} h_{b_1 a_2}) x^{a_1} x^{a_2} \partial_\alpha x^{b_1} \partial_\beta x^{b_2} \\ & + \lambda_1 (h_{a_1 a_2} x^{a_1} x^{a_2} g_{\alpha\beta} - h_{b_1 b_2} \partial_\alpha x^{b_1} \partial_\beta x^{b_2}) + 2\lambda_0 g_{\alpha\beta}, \end{aligned}$$

where the vector components x^i in V are regarded as functions on M by restriction.

For Euclidean 3-space a complete description of integrable Killing tensors has been obtained using computer algebra by Horwood, McLenaghan and Smirnov based on the prior knowledge of the separable coordinate webs, but a general solution of the integrability conditions has so far been considered intractable [HMS05]. Our algebraic equations now render this feasible at least in dimension three.[3] This goes beyond the scope of this article and will be the subject of a forthcoming paper [Sch].

In this context it is noteworthy that the first algebraic integrability condition can be recast into a variety of different forms. In terms of the curvature form $\Omega \in \mathrm{End}(V) \otimes \Lambda^2 V$ associated ot the algebraic curvature tensor R, condition (1.1a) reads

$$\Omega \wedge \Omega = 0,$$

where the wedge denotes the ususal exterior product in the form component and matrix multiplication in the endomorphism component. Another equivalent form, which makes more explicit the index symmetries in terms of $\mathrm{GL}(V)$-irrepresentations, is

$$\young(b_2b_1d_1,c_2,d_2,a_2) \quad g_{ij} R^i{}_{b_1 a_2 b_2} R^j{}_{d_1 c_2 d_2} = 0, \qquad (1.3)$$

where the operator on the left is the Young symmetriser antisymmetrising first in a_2, b_2, c_2, d_2 and then symmetrising in b_2, b_1, d_1. Similar forms can be obtained for the second algebraic integrability condition (1.1b).

The second and related advantage of our approach is, that the above algebraic formulation offers new insight into integrability from the perspectives of representation theory and algebraic geometry as well as geometric invariant theory. To illustrate this, regard the solutions of the first integrability condition as the algebraic variety given as the vanishing locus of the following composed map $\pi \circ \nu$ (where the spaces are denoted for convenience by their corresponding $\mathrm{GL}(V)$-isomorphism class):

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[2]Without loss of generality h can be supposed trace free.

[3]Note that dimension three of the manifold means dimension four of the ambient vector space.

$$\{\boxplus\} \quad \overset{\nu}{\longrightarrow} \quad \mathrm{Sym}^2\{\boxplus\} \quad \overset{\pi}{\longrightarrow} \quad \left\{\vcenter{\hbox{$\boxplus$}}\right\}$$

$$\nu\colon R_{a_1 b_1 a_2 b_2} \quad \mapsto \quad R_{a_1 b_1 a_2 b_2} R_{c_1 d_1 c_2 d_2}$$

$$\pi\colon T_{a_1 b_1 a_2 b_2 c_1 d_1 c_2 d_2} \quad \mapsto \quad \boxed{\begin{array}{ccc} b_2 & b_1 & d_1 \\ c_2 \\ d_2 \\ a_2 \end{array}} \ g^{a_1 c_1} T_{a_1 b_1 a_2 b_2 c_1 d_1 c_2 d_2} \,.$$

$$(1.4)$$

The first space is the space of algebraic curvature tensors on V and the second its symmetric product. The third space is the image of the Young symmetriser in $V^{\otimes 6}$. The map π is simply a projection given by an index contraction and a projection to an irreducible $\mathrm{GL}(V)$-representation, both commuting. If we pass to the projectivisation of $\pi \circ \nu$,

$$\mathbb{P}(\pi \circ \nu)\colon \ \mathbb{P}\{\boxplus\} \ \overset{\mathbb{P}\nu}{\longrightarrow} \ \mathbb{P}\,\mathrm{Sym}^2\{\boxplus\} \ \overset{\mathbb{P}\pi}{\longrightarrow} \ \mathbb{P}\left\{\vcenter{\hbox{$\boxplus$}}\right\},$$

then the map $\mathbb{P}\nu$ is nothing else than the Veronese embedding. This allows a geometric interpretation of the first integrability condition's (projectivised) solution space as the intersection of a Veronese variety with a certain projective subspace. The same is true for the second integrability condition. Of course, every projective variety is isomorphic to an intersection of a Veronese variety with a linear space [Har92], but here all spaces and maps are given explicitly.

Note also that in our algebraic setting an isometry invariant characterisation of the integrability of Killing tensors as in [HMS05] or [Hor07] reduces to chosing a suitable set of isometry invariants for algebraic curvature tensors and finding the restrictions imposed on them by the equations (1.1). This is essentially a problem in geometric invariant theory. Due to its importance in general relativity, a variety of such sets have already been proposed in the case of four-dimensional Lorentz space.[4]

Thirdly, we emphasise that our aproach is completely generic in the sense that is does not depend neither on the dimension of the manifold, nor the value of the constant sectional curvature nor the signature of the pseudo-Riemannian metric. This becomes manifest in the fact that these data enter the algebraic integrability conditions (1.1) only via the signature and the rank of $\bar{g}$. We also remark that our approach is coordinate free. We do not rely on any particular choice of coordinates neither on the manifold, nor on the space of Killing tensors.

Finally, a last but not less important advantage comes from the fact that our algebraic equations are polynomials in a curvature tensor. Note that we owe this circumstance to the fact that Killing tensors on constant sectional curvature manifolds are described by algebraic curvature tensors and not any other representation space of the isometry group. This is a rather fortunate happenstance, because algebraic properties of curvature tensors are extensively studied – both in differential geometry as well as in mathematical physics. Especially the Lorentzian case, focus of interest in general relativity, is important here as it corresponds to hyperbolic space. In the Riemannian case, corresponding to spheres, we even have a complete classification of the symmetry classes of the Riemann polynomials appearing in (1.1) with respect to the isometry group $\mathrm{O}(V,g) \subset \mathrm{GL}(V)$. Our methods are inspired by the corresponding article of Fulling et al. [FKWC92], although we do not rely on results presented there.

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[4]See [LC04] and the references therein.

We hope that our work will pave the way for an algebraic approach to the study of integrable Killing tensors and seperable coordinates. To this end we list some suggestions for future research based on our results:

- *Algebraic interpretation of other families of integrable Killing tensors,* such as the one arising from special conformal Killing tensors [CST00, CS01, Cra03a], cofactor systems [RWML99, Lun01, RW09] or bi-quasi-Hamiltonian systems [CS02, Cra03b] (see also [Ben05]). We do not yet fully understand how this – geometrically constructed – family translates to our algebraic framework, but we believe there is a simple algebraic interpretation. Vice versa, we neither know a geometric interpretation of the – algebraically constructed – family we constructed in the present work.
- *An algebraic compatibility condition for the potential.* So far we disregarded the compatibility condition for the potential in the Hamilton-Jacobi equation. As integrability, it should be expressible entirely in algebraic terms as well.
- *Explicit solution of the algebraic integrability conditions.* It is possible to solve the algebraic integrability conditions explicitly in dimension three. This has been done for 3-spheres in [Sch] and straightforwardly carries over to Euclidean 3-space. In higher dimensions they can be solved using computer algebra, by means of Gröbner bases for example.
- *Study of the algebraic variety of integrable Killing tensors,* defined by the algebraic integrability conditions, especially its dimension.

This paper is organised as follows. In the next section we briefly recall Killing tensors, constant sectional curvature manifolds and the notion of integrability in this context. In section 3 we regard a special family of integrable Killing tensors. This is followed by a short review of some necessary facts from the representation theory of symmetric and linear groups in section 4, which can be skipped by a reader familiar with them. After that we restate the algebraic characterisation of Killing tensors in section 5 for our needs. Section 6 is the main part, where we derive the algebraic integrability conditions. As a first application of them, we extend the family from section 3 in the last section.

2. Preliminaries

2.1. Killing tensors.
Recall that a *Killing vector* on a (pseudo-)Riemannian manifold (M, g) is a vector field K^α on M satisfying the Killing equation

$$\nabla^{(\alpha} K^{\beta)} = 0$$

where ∇ is Levi-Civita connection of g and round brackets denote complete symmetrisation in the enclosed indices.

Definition 2.1. *A* Killing tensor *on M is a symmetric $(2, 0)$-tensor field $K^{\alpha\beta}$ satisfying the generalised Killing equation*

$$\nabla^{(\alpha} K^{\beta\gamma)} = 0 \,. \tag{2.1}$$

Geometrically, a tensor $K^{\alpha\beta}$ is a Killing tensor if and only if the amount $K^{\alpha\beta} \dot{x}_\alpha \dot{x}_\beta$ is constant on geodesics.

Examples 2.2.

 (i) *The metric is a Killing tensor, since it is covariantly constant.*
 (ii) *As a consequence of the Leibniz rule the symmetrised tensor product of two Killing vectors is a Killing tensor.*
 (iii) *The pullback of a Killing tensor under an isometry of M is again a Killing tensor.*

2.2. **Integrability.** The metric establishes an isomorphism between the tangent space and its dual. This identifies co- and contravariant tensor components via lowering or rising indices using the metric. In particular a Killing tensor can be identified with a $(0,2)$-tensor or a $(1,1)$-tensor, the latter being an endomorphism field on M.

Definition 2.3. *A $(1,1)$-tensor field K on M is* integrable, *if almost every point on M admits a neighbourhood with local coordinates x_α such that the corresponding coordinate vector fields ∂_α are eigenvector fields of K.*

Integrability can be characterised using the Nijenhuis torsion

$$N(X,Y) := K^2[X,Y] - K[KX,Y] - K[X,KY] + [KX,KY],$$

given in local coordinates by

$$N^\alpha{}_{\beta\gamma} = K^\alpha{}_\delta \nabla_{[\gamma} K^\delta{}_{\beta]} + \nabla_\delta K^\alpha{}_{[\gamma} K^\delta{}_{\beta]},\tag{2.2}$$

where square brackets denote complete antisymmetrisation in the enclosed indices [Nij51]:

Theorem 2.4. *Let N be the Nijenhuis torsion of K. Then K is integrable if and only if the following conditions hold:*

$$0 = N^\delta{}_{[\beta\gamma} g_{\alpha]\delta}\tag{2.3a}$$

$$0 = N^\delta{}_{[\beta\gamma} K_{\alpha]\delta}\tag{2.3b}$$

$$0 = N^\delta{}_{[\beta\gamma} K_{\alpha]\varepsilon} K^\varepsilon{}_\delta\tag{2.3c}$$

2.3. **Manifolds with constant sectional curvature.** The Killing equation is linear, so the set of Killing tensors on M is a vector space. Its maximal dimension is

$$\frac{n(n+1)^2(n+2)}{12} \qquad\qquad n = \dim M \tag{2.4}$$

and will be attained if and only if M has constant sectional curvature [Tho86, Wol98]. Every (pseudo-)Riemannian manifold with constant sectional curvature is (up to a rescaling) locally isometric to one of the standard models below. This fact allows us to restrict all subsequent considerations to these standard models.

Examples 2.5 (Standard models of manifolds with constant sectional curvature)**.** *Let V be a vector space of dimension $N := n + 1$, equipped with a non-degenerate inner product g of signature (p, q). Then the (pseudo-)sphere*

$$M := \{x \in V : g(x, x) = 1\}$$

is an n-dimensional (pseudo-)Riemannian manifold of constant sectional curvature with respect to the metric obtained by restricting g to M. For $(p, q) = (n + 1, 0)$ this is the standard sphere and for $(p, q) = (n, 1)$ the standard hyperbolic space. For other choices of the signature we obtain the different de Sitter and anti de Sitter

spaces. The isometry group of M is the (pseudo-)orthogonal group $\mathrm{O}(V, g) \subset \mathrm{GL}(V)$ acting on M by restriction.

We can incorporate flat space in this pattern by embedding it as the hyperplane

$$M := \{x \in V : g(x, u) = 1\}$$

in V for some fixed normal vector $u \in V$. The corresponding isometry group is then embedded in $\mathrm{GL}(V)$ as the semi-direct product $M \rtimes \mathrm{O}(M, g)$. Although our approach goes through for flat spaces as well, this case often has to be treated seperately.

3. BENENTI TENSORS

For the time being let the inner product g be positive definite so that $M \subset V$ is the unit sphere. Consider the diffeomorphism

$$
\begin{array}{rccl}
f_A : & M & \to & M \\
& x & \mapsto & f_A(x) := \frac{Ax}{\|Ax\|}
\end{array}
\tag{3.1}
$$

for some fixed $A \in \mathrm{GL}(V)$. Since A is linear, f_A maps great circles to great circles. This means that the metric g and its pullback $g_A := f_A^* g$ have the same (unparametrised) geodesics, so we can apply the following theorem [MT98, MT00]:

Theorem 3.1. *If two metrics g and g_A on an n-dimensional (pseudo-)Riemannian manifold have the same unparametrised geodesics, then*

$$K := \left(\frac{\det g}{\det g_A} \right)^{\frac{2}{n+1}} g_A$$

is a Killing tensor with respect to g.

Corollary 3.2. *Let $M \subset V$ be the unit sphere. Then the group $\mathrm{GL}(V)$ generates a family of Killing tensors on M given by*

$$K_x(v, w) = g(Ax, Ax)g(Av, Aw) - g(Ax, Av)g(Ax, Aw) \tag{3.2}$$

for $v, w \in T_x M$ and $A \in \mathrm{GL}(V)$.

Proof. Via the differential of (3.1),

$$(df_A)_x v = \frac{g(Ax, Ax)Av - g(Ax, Av)Ax}{\|Ax\|^3}, \tag{3.3}$$

one computes the pullback metric $g_A = f_A^* g$ as

$$(g_A)_x(v, w) = \frac{g(Ax, Ax)g(Av, Aw) - g(Ax, Av)g(Ax, Aw)}{\|Ax\|^4}.$$

To compute its determinant at a point x, choose an orthonormal basis $e_1, \ldots, e_n$ of $T_x M$ and extend it with the vector x to an orthonormal basis of V. Since $Ax/\|Ax\|$ is a unit vector normal to $T_{f_A(x)} M$, we have

$$\left(\frac{\det g_A}{\det g} \right)_x = \det^2 (df_A)_x = \det^2 \left(\frac{Ax}{\|Ax\|}, (df_A)_x e_1, \ldots, (df_A)_x e_n \right)$$

$$= \det^2 \left(\frac{Ax}{\|Ax\|}, \frac{Ae_1}{\|Ax\|}, \ldots, \frac{Ae_n}{\|Ax\|} \right) = \left(\frac{\det A}{\|Ax\|^{n+1}} \right)^2,$$

where we used (3.3) for the third equality. The claim now follows from theorem 3.1, since

$$\left(\frac{\det g}{\det g_A}\right)^{\frac{2}{n+1}} g_A = \left(\frac{\|Ax\|^{n+1}}{\det A}\right)^{\frac{4}{n+1}} \frac{g(Ax, Ax)g(Av, Aw) - g(Ax, Av)g(Ax, Aw)}{\|Ax\|^4},$$

differs from (3.2) by a constant.   □

If g is not positive definite, the map (3.1) is not everywhere well defined, but formula (3.2) still gives a well defined Killing tensor, as we will see in section 5. For flat space the result is analogous, only with more complicated expressions.

Corollary 3.3. *Let $M \subset V$ be a hyperplane with normal u. Then the group $\mathrm{GL}(V)$ generates a family of Killing tensors on M given by*

$$K_x(v, w) = g(Ax, u)g(Ax, u)g(Av, Aw) - g(Ax, u)g(Ax, Av)g(Aw, u)$$
$$- g(Ax, u)g(Ax, Aw)g(Av, u) + g(Ax, Ax)g(Av, u)g(Aw, u). \quad (3.4)$$

Proof. The proof follows the lines of the proof in the non-flat case, considering the diffeomorphism

$$f_A \colon \quad M \quad \to \quad M$$
$$x \quad \mapsto \quad f_A(x) := \frac{Ax}{g(u, Ax)} \quad (3.5)$$

instead of (3.1). This map is line preserving, so we can again apply theorem 3.1. The differential

$$(df_A)_x v = \frac{g(u, Ax)Av - g(u, Av)Ax}{g(u, Ax)^2}$$

of (3.5) yields the pullback metric

$$(g_A)_x(v, w)$$
$$= \frac{1}{g(u, Ax)^4}\Big[g(u, Ax)g(u, Ax)g(Av, Aw) - g(u, Av)g(u, Ax)g(Ax, Aw)$$
$$- g(u, Ax)g(u, Aw)g(Av, Ax) + g(u, Av)g(u, Aw)g(Ax, Ax)\Big].$$

As above, we compute the determinant of $(g_A)_x$ using an othonormal basis $e_1, \ldots, e_n$ of $T_x M$,

$$\left(\frac{\det g_A}{\det g}\right)_x = \det^2(df)_x = \det^2\big(u, (df)_x e_1, \ldots, (df)_x e_n\big)$$
$$= \det^2\big(\tfrac{Ax}{g(Ax,u)}, (df)_x e_1, \ldots, (df)_x e_n\big)$$
$$= \det^2\left(\tfrac{Ax}{g(Ax,u)}, \tfrac{Ae_1}{g(Ax,u)}, \ldots, \tfrac{Ae_n}{g(Ax,u)}\right)$$
$$= \left(\frac{\det A}{g(Ax, u)^{n+1}} \det\big(x, e_1, \ldots, e_n\big)\right)^2 = \left(\frac{\det A}{g(Ax, u)^{n+1}}\right)^2,$$

and the corollary follows from theorem 3.1.   □

Killing tensors of type (3.2) respectively (3.4) coincide with those introduced in local coordinates in [Ben92]. Following [IMM00, BM03] we will call them *Benenti tensors*.

4. SOME FACTS FROM REPRESENTATION THEORY

We briefly collect some facts we need from the representation theory of symmetric and general linear groups. More details can be found in standard textbooks.

4.1. **Young symmetrisers and the Littlewood-Richardson rule.** The isomorphism classes of irreducible representations ("irreps") of S_d are labelled by *partitions* of d, i.e. integers $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_r > 0$ with $d_1 + \ldots + d_r = d$. It is useful to depict partitions as so called *Young frames*, as in the following example:

Example 4.1. *The partitions of* 4 *are* $4 = 4$, $3 + 1 = 4$, $2 + 2 = 4$, $2 + 1 + 1 = 4$ *and* $1 + 1 + 1 + 1 = 4$ *with corresponding Young frames*
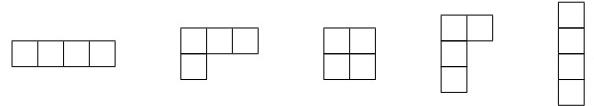
The dimension of an irrep corresponding to a Young frame is given by dividing $d!$ by the product of the hook lengths of all boxes of the frame, where the hook length of a box is

(the number of boxes to the right) $+$ 1 $+$ (the number of boxes below).

This is the so called *hook formula*.

Example 4.2. *The hook lengths of the boxes of the Young frames in example 4.1 are*
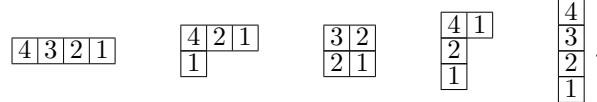
The corresponding dimensions are thus 1, 3, 2, 3, 1.

The irreps of S_d can be realised on subspaces of the *group algebra*

$$\mathbb{R}\,S_d = \left\{ \sum_{\pi \in S_d} \lambda_\pi \pi \colon \lambda_\pi \in \mathbb{R} \right\} .$$

This is the free real vector space over S_d as a set, endowed with the obvious multiplication given by extending the group multiplication in S_d linearly. Multiplication with elements of S_d from the left defines a representation of S_d on $\mathbb{R}\,S_d$. One then constructs projectors onto irreducible subspaces as follows.

Let τ be a Young frame whose d boxes are filled with the integers $1, \ldots, d$ without repetition in an arbitrary order. We call this a *Young tableau*. For each row r of τ let $S_r \subset S_d$ be the symmetric group of permutations of the labels in r and similarly $S_c \subset S_d$ for each column c of τ. The *Young symmetriser* corresponding to τ is then the element in $\mathbb{R}\,S_d$ defined by

$$\prod_{r \text{ row of } \tau} \underbrace{\left(\sum_{\pi \in S_r} \pi \right)}_{\text{row symmetriser of } r} \quad \prod_{c \text{ column of } \tau} \underbrace{\left(\sum_{\pi \in S_c} (\operatorname{sign} \pi)\pi \right)}_{\text{column antisymmetriser of } c} \quad . \tag{4.1}$$

We will often identify a Young tableau with its corresponding Young symmetriser as in the following example.

Example 4.3.

$$\young(12,34) = \young(12)\;\young(34)\;\young(1,3)\;\young(2,4) = \big(e + (12)\big)\big(e + (34)\big)\big(e - (13)\big)\big(e - (24)\big) \tag{4.2}$$

$$= e + (12) + (34) - (13) - (24) + (12)(34) + (13)(24) - (132) - (234)$$

$$- (124) - (143) + (1423) + (1324) - (1234) - (1432) + (14)(23)$$

If we denote by h_τ the product of the hook lengths of all boxes in a Young tableau τ, the corresponding Young symmetriser satisfies

$$\tau^2 = h_\tau \tau. \tag{4.3}$$

Every element in $\mathbb{R}\,\mathrm{S}_d$ is at the same time a linear operator on $\mathbb{R}\,\mathrm{S}_d$ via multiplication from the right. Then the image of τ is an irreducible subrepresentation of $\mathbb{R}\,\mathrm{S}_d$ whose isomorphism class is given by the Young frame of τ. Rewriting (4.3) we get a projector onto the corresponding subspace:

$$\left(\frac{\tau}{h_\tau}\right)^2 = \frac{\tau}{h_\tau}. \tag{4.4}$$

The group algebra $\mathbb{R}\,\mathrm{S}_d$ carries a natural S_d-invariant inner product defined by taking the basis S_d to be orthonormal. With respect to this inner product the adjoint of an element in $\mathbb{R}\,\mathrm{S}_d$ is

$$\left(\sum_{\pi \in \mathrm{S}_d} \alpha_\pi \pi\right)^\star = \sum_{\pi \in \mathrm{S}_d} \alpha_\pi \pi^{-1}.$$

Note that the column symmetrisers of a Young tableau are self-adjoint and commute and likewise for its row antisymmetrisers. Taking the adjoint of a Young symmetriser therefore simply exchanges the two products in (4.1).

Example 4.4.

$$\young(12,34)^\star = \young(1,3)\;\young(2,4)\;\young(12)\;\young(34) = \big(e - (13)\big)\big(e - (24)\big)\big(e + (12)\big)\big(e + (34)\big) \tag{4.5}$$

We see that Young symmetrisers are in general not self-adjoint, so that the corresponding Young projectors (4.4) will not be orthogonal. However, a Young projector and its adjoint project onto isomorphic irreps [Ful97]. From (4.1) and (4.3) follows that up to an apropriate factor the element $\tau\tau^\star$ is an orthogonal projector onto the image of τ and similarly $\tau^\star\tau$ onto the image of $\tau^\star$.

Two representations λ_1 and λ_2 of S_{d_1} on V_1 respectively S_{d_2} on V_2 determine a representation of $\mathrm{S}_{d_1} \times \mathrm{S}_{d_2}$ on $V_1 \otimes V_2$ given by

$$(g_1 \times g_2)(v_1 \otimes v_2) := (g_1 v_1) \otimes (g_2 v_2) \qquad g_1 \in \mathrm{S}_{d_1},\ g_2 \in \mathrm{S}_{d_2}.$$

Via the inclusion $\mathrm{S}_{d_1} \times \mathrm{S}_{d_2} \hookrightarrow \mathrm{S}_{d_1+d_2}$ this induces a representation $\lambda_1 \boxtimes \lambda_2$ of $\mathrm{S}_{d_1+d_2}$ on $V_1 \otimes V_2$, called the exterior tensor product of λ_1 and λ_2. The Littlewood-Richardson rule tells us how this product decomposes into irreps:

Theorem 4.5 (The Littlewood-Richardson rule)**.** *The decomposition of the exterior tensor product $\lambda_1 \boxtimes \lambda_2$ of two irreps λ_1 of S_{d_1} and λ_2 of S_{d_2} into irreps of $S_{d_1+d_2}$ is given by the following algorithm. First label all the boxes in λ_2 with their corresponding row number. Then add the labelled boxes of λ_2 to λ_1 – one by one from top to bottom – respecting at each step the following rules:*

(i) *The obtained frame is a legitimate Young frame.*
(ii) *No two boxes in the same column are labelled equally.*
(iii) *If the labels are read off from right to left along the rows from top to bottom, one never encounters more 1's than 2's, and so on.*

Each of the distinct Young frames constructed in this way specifies an irreducible sum term in the decomposition of $\lambda_1 \boxtimes \lambda_2$ with the correpsonding multiplicity, since the same shaped Young frame may arise in more than one way. Since the exterior tensor product is commutative, one can choose the simpler Young frame for λ_2.

Example 4.6.

$$\Box\Box \boxtimes \boxed{1\,1} \cong \Box\Box\boxed{1\,1} \oplus \begin{array}{c}\boxed{1}\\\boxed{1}\end{array} \oplus \boxed{1\,1} \tag{4.6}$$

$$\begin{array}{c}\Box\\\Box\end{array} \boxtimes \begin{array}{c}\boxed{1}\\\boxed{2}\end{array} \cong \begin{array}{c}\boxed{1}\\\boxed{2}\end{array} \oplus \begin{array}{c}\boxed{1}\\\boxed{2}\end{array} \oplus \begin{array}{c}\boxed{1}\\\boxed{2}\end{array} \tag{4.7}$$

$$\begin{array}{c}\Box\\\Box\end{array} \boxtimes \boxed{1\,1\,1} \cong \boxed{1\,1\,1} \oplus \begin{array}{c}\boxed{1\,1}\\\boxed{1}\end{array} \tag{4.8}$$

4.2. **Weyl's construction and algebraic curvature tensors.** Every Young tableau τ gives rise to a $\mathrm{GL}(V)$-irrep in the following way, called Weyl's construction. Consider the d-fold tensor product $V^{\otimes d}$ as a representation space for both $\mathrm{GL}(V)$ and S_d with respect to the commuting actions

$$g(v_{i_1} \otimes \ldots \otimes v_{i_d}) := (gv_{i_1}) \otimes \ldots \otimes (gv_{i_d}) \qquad g \in \mathrm{GL}(V)$$
$$\pi(v_1 \otimes \ldots \otimes v_d) := v_{\pi^{-1}(1)} \otimes \ldots \otimes v_{\pi^{-1}(d)} \qquad \pi \in \mathrm{S}_d .$$

Each element in $\mathbb{R}\,\mathrm{S}_d$ gives a linear operator on $V^{\otimes d}$ by linearly extending the action of S_d. The image of a Young symmetriser $\tau \in \mathbb{R}\,\mathrm{S}_d$ is then an irreducible $\mathrm{GL}(V)$-subrepresentation. Considering instead the dual action of $\mathrm{GL}(V)$ on the dual $\bar{V}$ of V yields the (non-isomorphic) dual representation on $\bar{V}^{\otimes d}$.

Example 4.7. *The Young symmetriser* (4.2) *determines the following operator on $\bar{V}^{\otimes 4}$ whose image constitutes a $\mathrm{GL}(V)$-irrep:*

$$\begin{array}{|c|c|}\hline a_1 & a_2\\\hline b_1 & b_2\\\hline\end{array} \; T_{a_1 a_2 b_1 b_2} = T_{a_1 a_2 b_1 b_2} - T_{a_2 a_1 b_1 b_2} - T_{a_1 a_2 b_2 b_1} + T_{a_2 a_1 b_2 b_1}$$
$$+ T_{b_1 a_2 a_1 b_2} - T_{a_2 b_1 a_1 b_2} - T_{b_1 a_2 b_2 a_1} + T_{a_2 b_1 b_2 a_1}$$
$$+ T_{a_1 b_2 b_1 a_2} - T_{b_2 a_1 b_1 a_2} - T_{a_1 b_2 a_2 b_1} + T_{b_2 a_1 a_2 b_1}$$
$$+ T_{b_1 b_2 a_1 a_2} - T_{b_2 b_1 a_1 a_2} - T_{b_1 b_2 a_2 a_1} + T_{b_2 b_1 a_2 a_1} . \tag{4.9}$$

Note that on the level of tensor components one gets the correct action of S_d by permuting index names, not index positions.

In the same way we can construct an irreducible $\mathrm{GL}(V)$-subrepresentation from the adjoint $\tau^\star$ of a Young tableau τ.

Example 4.8.

$$\boxed{\begin{smallmatrix}a_1 & a_2 \\ b_1 & b_2\end{smallmatrix}}^{\star} T_{a_1b_1a_2b_2} = T_{a_1b_1a_2b_2} + T_{a_2b_1a_1b_2} + T_{a_1b_2a_2b_1} + T_{a_2b_2a_1b_1}$$
$$- T_{b_1a_1a_2b_2} - T_{a_2a_1b_1b_2} - T_{b_1b_2a_2a_1} - T_{a_2b_2b_1a_1}$$
$$- T_{a_1b_1b_2a_2} - T_{b_2b_1a_1a_2} - T_{a_1a_2b_2b_1} - T_{b_2a_2a_1b_1} \tag{4.10}$$
$$+ T_{b_1a_1b_2a_2} + T_{b_2a_1b_1a_2} + T_{b_1a_2b_2a_1} + T_{b_2a_2b_1a_1} .$$

Example 4.9 (Algebraic curvature tensors). *An algebraic curvature tensor on V is an element $R \in \bar{V}^{\otimes 4}$ satisfying the symmetry relations of a Riemannian curvature tensor, i.e.:*

$$\text{antisymmetry:} \quad R_{b_1a_1a_2b_2} = -R_{a_1b_1a_2b_2} = R_{a_1b_1b_2a_2} \tag{4.11a}$$
$$\text{pair symmetry:} \quad R_{a_2b_2a_1b_1} = R_{a_1b_1a_2b_2} \tag{4.11b}$$
$$\text{Bianchi identity:} \quad R_{a_1b_1a_2b_2} + R_{a_1a_2b_2b_1} + R_{a_1b_2b_1a_2} = 0 \tag{4.11c}$$

A little computation shows, that on one hand any tensor of the form (4.10) *has these symmetries and that on the other hand any tensor having these symmetries verifies*

$$\frac{1}{12}\boxed{\begin{smallmatrix}a_1 & a_2 \\ b_1 & b_2\end{smallmatrix}}^{\star} R_{a_1b_1a_2b_2} = R_{a_1b_1a_2b_2} .$$

This means that algebraic curvature tensors form an irreducible $\mathrm{GL}(V)$*-representation space.*

Example 4.10 (Symmetrised algebraic curvature tensors). *A symmetrised algebraic curvature tensor on V is an element $S \in \bar{V}^{\otimes 4}$ satisfying the following symmetry relations:*

$$\text{symmetry:} \quad S_{a_2a_1b_1b_2} = +S_{a_1a_2b_1b_2} = S_{a_1a_2b_2b_1} \tag{4.12a}$$
$$\text{pair symmetry:} \quad S_{b_1b_2a_1a_2} = S_{a_1a_2b_1b_2} \tag{4.12b}$$
$$\text{Bianchi identity:} \quad S_{a_1a_2b_1b_2} + S_{a_1b_2a_2b_1} + S_{a_1b_1b_2a_2} = 0 \tag{4.12c}$$

As in the previous example, this is equivalent to

$$\frac{1}{12}\boxed{\begin{smallmatrix}a_1 & a_2 \\ b_1 & b_2\end{smallmatrix}} S_{a_1a_2b_1b_2} = S_{a_1a_2b_1b_2}$$

so that symmetrised algebraic curvature tensors form another irreducible $\mathrm{GL}(V)$*-representation space.*

Remark 4.11 (Bianchi identity). *In presence of the first two symmetries, there are several equivalent forms of the Bianchi identity in both cases. First, we can write it as vanishing cyclic sum over* any *three of the four indices, for example as $R_{a_1b_1a_2b_2} + R_{b_1a_2a_1b_2} + R_{a_2a_1b_1b_2} = 0$. Second, for (symmetrised) algebraic curvature tensors the Bianchi identity is equivalent to the vanishing of the complete antisymmetrisation (symmetrisation) in any three of the four indices, for example to*

$$\boxed{\begin{smallmatrix}b_1 \\ a_2 \\ b_2\end{smallmatrix}} R_{a_1b_1a_2b_2} = 0$$

or

$$\boxed{\begin{smallmatrix}a_2 & b_1 & b_2\end{smallmatrix}} S_{a_1a_2b_1b_2} = 0 . \tag{4.13}$$

In the following we will refer to all these forms as "Bianchi identity".

The GL(V)-irreps constructed from τ and $\tau^\star$ are isomorphic.

Example 4.12. *An explicit isomorphism between the irreps of* GL(V) *on algebraic curvature tensors respectively on symmetrised algebraic curvature tensors is given by*

$$S_{a_1 a_2 b_1 b_2} = \frac{1}{\sqrt{3}}\left(R_{a_1 b_1 a_2 b_2} + R_{a_1 b_2 a_2 b_1}\right) \tag{4.14a}$$

$$R_{a_1 b_1 a_2 b_2} = \frac{1}{\sqrt{3}}\left(S_{a_1 a_2 b_1 b_2} - S_{a_1 b_2 b_1 a_2}\right), \tag{4.14b}$$

which is easily checked using the symmetries (4.11) *and* (4.12).

Two Young tableaux determine isomorphic GL(V)-representations if and only if they fill the same Young frame λ. Their dimension can be computed by labelling each box of λ with

(number of the box' column) $+$ N $-$ (number of the box' row),

taking the product of these labels and dividing by the product h_λ of the hook lengths of λ. It is standard to denote the isomorphism class obtained from λ via the Weyl construcction by $\{\lambda\}$.

Example 4.13. *The isomorphism class of the irrepresentations given by (symmetrised) algebraic curvature tensors is* $\{\boxplus\}$ *an has dimension*

$$\dim\left\{\boxplus\right\} = \frac{(N-1)N^2(N+1)}{12}. \tag{4.15}$$

The dual pairing between $\bar{V}^{\otimes d}$ and $V^{\otimes d}$ is given by index contraction. From the identity

$$S_{i_1 \cdots i_d} T^{i_{\pi(1)} \cdots i_{\pi(d)}} = S_{i_{\pi^{-1}(1)} \cdots i_{\pi^{-1}(d)}} T^{i_1 \cdots i_d}$$

we deduce

$$S_{i_1 \cdots i_d}\left(\pi T^{i_1 \cdots i_d}\right) = \left(\pi^{-1} S_{i_1 \cdots i_d}\right) T^{i_1 \cdots i_d} = \left(\pi^\star S_{i_1 \cdots i_d}\right) T^{i_1 \cdots i_d}.$$

This means that with respect to the dual pairing the adjoint of the linear operator on $V^{\otimes d}$ given by an element $\tau \in \mathbb{R}\,\mathrm{S}_d$ acting on upper indices is given by $\tau^\star$ acting on lower indices. To save notation we will use parentheses as above to indicate whether a given element of $\mathbb{R}\,\mathrm{S}_d$ acts on upper or lower indices.

5. An algebraic characterisation of Killing tensors

Recall that we consider standard models of constant sectional curvature manifolds M, embedded isometrically as hypersurfaces in a Euclidean vector space (V, g). As common in relativity, we distinguish coordinates on M and V by index types:

Convention 5.1. *Throughout this exposition we use latin indices $a, b, c, \ldots$ for components in V (ranging from 0 to n) and greek indices $\alpha, \beta, \gamma, \ldots$ for local coordinates on M (ranging from 1 to n). We can then denote both the inner product on V as well as the induced metric on M by the same letter g and distinguish both via the type of indices. Consequently, latin indices are rised and lowered using g_{ab} and greek ones using $g_{\alpha\beta}$.*

The key result for our algebraic characterisation of integrability stems from McLenaghan, Milson & Smirnov and is a special case of theorem 3.5 in [MMS04]:

Theorem 5.2. *Let $M \subset V$ be one of the standard models for constant sectional curvature manifolds as in example 2.5.*

(i) *There is an isomorphism between the irreducible $\mathrm{GL}(V)$-representation space of antisymmetric tensors A_{ab} on V and the vector space of Killing vectors K on M, given by*

$$K_x(v) := A_{ab}x^a v^b \qquad\qquad x \in M, \ v \in T_x M\,,$$

when K is written covariantly.

(ii) *There is an isomorphism between the irreducible $\mathrm{GL}(V)$-representation space of algebraic curvature tensors $R_{a_1 b_1 a_2 b_2}$ on V and the vector space of Killing tensors K on M, given by*

$$K_x(v,w) := R_{a_1 b_1 a_2 b_2} x^{a_1} v^{b_1} x^{a_2} w^{b_2} \qquad x \in M, \ v,w \in T_x M\,, \qquad (5.1)$$

when K is written covariantly.

Both isomorphisms are equivariant with respect to the action of the isometry group of M as a subgroup of $\mathrm{GL}(V)$.

First note that due to the term $x^{a_1} x^{a_2}$ the tensor $R_{a_1 b_1 a_2 b_2}$ in (5.1) is implicitly symmetrised in the indices a_1, a_2 and can therefore be replaced by the corresponding symmetrised algebraic curvature tensor (4.14a). Since this will simplify subsequent computations considerably, we reformulate the the second part of the theorem:

Corollary 5.3. *Let $M \subset V$ be one of the standard models as in example 2.5. Then*

$$K(v,w) := S_{a_1 a_2 b_1 b_2} x^{a_1} x^{a_2} v^{b_1} w^{b_2} \qquad\qquad x \in M, \ v,w \in T_x M \qquad (5.2)$$

defines an isomorphism between the irreducible $\mathrm{GL}(V)$-representation space of symmetrised algebraic curvature tensors $S_{a_1 a_2 b_1 b_2}$ and the vector space of Killing tensors K on M, which is equivariant with respect to the action of the isometry group.

We include a short proof here, because some ideas and intermediate results will be useful in subsequent computations.

Remark 5.4. *If we consider the standard coordinates x^a of a vector $x \in V$ as functions on $M \subset V$ by restriction, then we can write for any tangent vector $u \in T_x M \subset V$ with coordinates u^a in V:*

$$\nabla_u x^a = u^a\,, \qquad\qquad (5.3)$$

where ∇ denotes the Levi-Civita connection of the metric on M.

Proof (of corollary 5.3). We first show that the tensor (5.2) actually is a Killing tensor. Extend the vectors $u,v,w \in T_x M$ to arbitrary vector fields $\bar{u}, \bar{v}, \bar{w}$ on M. Using (5.2) and (5.3), we compute

$$\begin{aligned}
\nabla_u K(v,w) &= \nabla_{\bar{u}}\big(K(\bar{v},\bar{w})\big) - K\big(\nabla_{\bar{u}}\bar{v}, \bar{w}\big) - K\big(\bar{v}, \nabla_{\bar{u}}\bar{w}\big) \\
&= S_{a_1 a_2 b_1 b_2} \Big(u^{a_1} x^{a_2} v^{b_1} w^{b_2} + x^{a_1} u^{a_2} v^{b_1} w^{b_2} \\
&\qquad\qquad + x^{a_1} x^{a_2}(\nabla_{\bar{u}}\nabla_{\bar{v}} - \nabla_{\nabla_{\bar{u}}\bar{v}})x^{b_1} w^{b_2} \\
&\qquad\qquad + x^{a_1} x^{a_2} v^{b_1}(\nabla_{\bar{u}}\nabla_{\bar{w}} - \nabla_{\nabla_{\bar{u}}\bar{w}})x^{b_2} \Big)\,.
\end{aligned} \qquad (5.4)$$

The operator $H(u,v) = \nabla_{\bar{u}}\nabla_{\bar{v}} - \nabla_{\nabla_{\bar{u}}\bar{v}}$ is the Hesse operator and does not depend on the extensions $\bar{u}$ and $\bar{v}$ of u and v.

Lemma 5.5. *The Hesse form of the function x^b is given by*

$$H(u,v)x^b = -g(v,w)x^b$$

if M is not flat and zero otherwise.

Proof. Extend the vector fields $\bar{u}, \bar{v}$ on M further to all of V and denote the Levi-Civita connection in V by $\bar{\nabla}$. Then, using (5.3),

$$H(u,v)x^a = \nabla_{\bar{u}}\nabla_{\bar{v}}x^a - \nabla_{\nabla_{\bar{u}}\bar{v}}x^a = \nabla_{\bar{u}}(\bar{v}^a) - \left(\nabla_{\bar{u}}\bar{v}\right)^a = \bar{\nabla}_{\bar{u}}(\bar{v}^a) - \left(\nabla_{\bar{u}}\bar{v}\right)^a$$
$$= \left[\bar{\nabla}_{\bar{u}}\bar{v} - \nabla_{\bar{u}}\bar{v}\right]^a = \left[\mathrm{II}(u,v)\right]^a$$

It is not difficult to show that the second fundamental form of $M \subset V$ is $\mathrm{II}_x(u,v) = -g(u,v)x$ if M is not flat. Otherwise the lemma is trivial. $\square$

We resume the proof of corollary 5.3. Together with the Bianchi identity the lemma shows that the terms in (5.4) containing the Hesse form $H(u,v)x^b$ vanish. Using the symmetry of $S_{a_1a_2b_1b_2}$ in a_1, a_2 we get

$$\nabla_u K(v,w) = 2S_{a_1a_2b_1b_2}x^{a_1}u^{a_2}v^{b_1}w^{b_2} .$$

We reformulate the results obtained so far in local coordinates, using (5.3) again:

$$K_{\alpha\beta} = S_{a_1a_2b_1b_2}x^{a_1}x^{a_2}\nabla_\alpha x^{b_1}\nabla_\beta x^{b_2} \tag{5.5a}$$

$$\nabla_\gamma K_{\alpha\beta} = 2S_{c_1c_2d_1d_2}x^{c_1}\nabla_\gamma x^{c_2}\nabla_\alpha x^{d_1}\nabla_\beta x^{d_2} \tag{5.5b}$$

That K satisfies the Killing equation (2.1) is now a direct consequence of the Bianchi identity.

We continue the proof by showing that the map defined by (5.2) is injective. Suppose

$$S_{a_1a_2b_1b_2}x^{a_1}x^{a_2}v^{b_1}w^{b_2} = 0 \tag{5.6}$$

for all $x, v, w \in V$ with $x \in M$ and $v, w \in T_xM$. From the Bianchi identity we see that (5.6) is trivially satisfied if $v = x$ or $w = x$. We can thus drop the condition $v, w \in T_xM$ by decomposing $v, w \in V$ according to the splitting $V = T_xM \oplus \mathbb{R}x$. We can also drop the condition $x \in M$, since $\mathbb{R}M$ is dense in V. Finally, by polarisation we obtain $S_{a_1a_2b_1b_2}x^{a_1}y^{a_2}v^{b_1}w^{b_2} = 0$ for all $x, y, v, w \in V$ which is equivalent to $S = 0$.

Bijectivity now follows from the fact that the dimensions (4.15) and (2.4) of both spaces coincide for $N = n + 1$. Equivariance is evident. $\square$

Definition 5.6. *The* Kulkarni-Nomizu product $h \owedge k$ *of two symmetric tensors h and k is the algebraic curvature tensor*

$$(h \owedge k)_{a_1b_1a_2b_2} := h_{a_1a_2}k_{b_1b_2} - h_{a_1b_2}k_{b_1a_2} - h_{b_1a_2}k_{a_1b_2} + h_{b_1b_2}k_{a_1a_2}$$
$$= \frac{1}{4}\begin{array}{|c|c|}\hline a_1 & a_2 \\\hline b_1 & b_2 \\\hline\end{array}^{\star} h_{a_1a_1}k_{b_1b_2} .$$

In the language of representation theory this product corresponds to the projection of (4.6) to the $\boxplus$-component.

Example 5.7 (The metric)**.** *If M is not flat, the metric as a Killing tensor on M is represented by the algebraic curvature tensor $\frac{1}{2}g \owedge g$. This follows from (5.1) and*

$$\left(\tfrac{1}{2}g \owedge g\right)_{a_1b_1a_2b_2} = \frac{1}{8}\begin{array}{|c|c|}\hline a_1 & a_2 \\\hline b_1 & b_2 \\\hline\end{array}^{\star} g_{a_1a_2}g_{b_1b_2} = g_{a_1a_2}g_{b_1b_2} - g_{a_1b_2}g_{a_2b_1} , \tag{5.7}$$

since $g_{a_1a_2}x^{a_1}x^{a_2} = 1$ and $g_{a_1b_2}x^{a_1}v^{b_2} = 0$.

In the flat case the metric is represented by $(u \otimes u) \oslash g$, given by

$$\left((u \otimes u) \oslash g\right)_{a_1 b_1 a_2 b_2} = \frac{1}{4} \boxed{\begin{smallmatrix} a_1 & a_2 \\ b_1 & b_2 \end{smallmatrix}}^\star u_{a_1} u_{a_2} g_{b_1 b_2}$$
$$= u_{a_1} u_{a_2} g_{b_1 b_2} - u_{a_1} u_{b_2} g_{b_1 a_2} - u_{b_1} u_{a_2} g_{a_1 b_2} - u_{b_1} u_{b_2} g_{a_1 a_2} \,,$$
$$(5.8)$$

since $u_a x^a = 1$ and $u_b v^b = 0$.

The isometry group of M is a subgroup of $\mathrm{GL}(V)$. As a consequence of theorem 5.2 its action on the space of Killing tensors extends to a natural action of $\mathrm{GL}(V)$.

Example 5.8 (Benenti tensors). *Rewriting (3.2) respectively (3.4) in the form (5.1) shows that Benenti tensors are represented by the algebraic curvature tensors*

$$\tfrac{1}{2}(Ag) \oslash (Ag)$$

respectively

$$(Au \otimes Au) \oslash Ag$$

if M is flat. Here Ag denotes the image of g under the action of $A \in \mathrm{GL}(V)$ on symmetric tensors on V. We can interpret this by saying that Benenti tensors form the orbit of the metric under the natural action of $\mathrm{GL}(V)$ on Killing tensors.

6. The algebraic integrability conditions

We saw that Killing tensors on a constant sectional curvature manifold correspond to algebraic curvature tensors. The aim of this section is to translate the Nijenhuis integrability conditions for such Killing tensors into algebraic integrability conditions on the corresponding algebraic curvature tensors.

First note that in the integrability conditions (2.3) the Nijenhuis torsion (2.2) appears only antisymmetrised in its two lower indices β, γ. To simplify computations we will thus replace the Nijenhuis torsion $N^\alpha{}_{\beta\gamma}$ in the integrability conditions by the tensor

$$\bar{N}^\alpha{}_{\beta\gamma} := \tfrac{1}{2}\left(K^\alpha{}_\delta \nabla_\gamma K^\delta{}_\beta + K^\delta{}_\beta \nabla_\delta K^\alpha{}_\gamma\right) \qquad\qquad \bar{N}^\alpha{}_{[\beta\gamma]} = N^\alpha{}_{\beta\gamma} \,.$$

Together with (5.5) this can be written as

$$\bar{N}^\alpha{}_{\beta\gamma} = S_{a_1 a_2 b_1 b_2} S_{c_1 c_2 d_1 d_2} x^{a_1} x^{a_2} x^{c_1} \nabla^\alpha x^{b_1} \nabla_\delta x^{b_2} \nabla_\gamma x^{c_2} \nabla^\delta x^{d_1} \nabla_\beta x^{d_2}$$
$$+ S_{a_1 a_2 b_1 b_2} S_{c_1 c_2 d_1 d_2} x^{a_1} x^{a_2} x^{c_1} \nabla^\delta x^{b_1} \nabla_\beta x^{b_2} \nabla_\delta x^{c_2} \nabla^\alpha x^{d_1} \nabla_\gamma x^{d_2} \,.$$

Lemma 6.1. *Let M be one of the standard models for constant sectional curvature manifolds as in example 2.5. Then*

$$\nabla_\delta x^a \nabla^\delta x^b = \begin{cases} g^{ab} - u^a u^b & \text{if } M \text{ is flat} \\ g^{ab} - x^a x^b & \text{otherwise}. \end{cases}$$

Proof. Let $e_1, \ldots, e_n$ be a basis of $T_x M$ and complete it with a unit normal vector $e_0 := u$ to a basis of V. Then on one hand

$$\sum_{i,j=0}^{n} g(e^i, e^j) \nabla_{e_i} x^a \nabla_{e_j} x^b = \sum_{i,j=1}^{n} g(e^i, e^j) \nabla_{e_i} x^a \nabla_{e_j} x^b + \nabla_u x^a \nabla_u x^b$$
$$= g^{\alpha\beta} \nabla_\alpha x^a \nabla_\beta x^b + u^a u^b \,.$$

On the other hand, choosing the standard basis of V instead, the left hand side is just g^{ab}. This proves the lemma, remarking that we can choose $u = x$ if M is not flat. $\qquad\square$

For flat M the lemma yields

$$
\begin{aligned}
\bar{N}^{\alpha}{}_{\beta\gamma} &= \bar{g}^{b_2 d_1} S_{a_1 a_2 b_1 b_2} S_{c_1 c_2 d_1 d_2} x^{a_1} x^{a_2} x^{c_1} \nabla^{\alpha} x^{b_1} \nabla_{\beta} x^{d_2} \nabla_{\gamma} x^{c_2} \\
&+ \bar{g}^{b_1 c_2} S_{a_1 a_2 b_1 b_2} S_{c_1 c_2 d_1 d_2} x^{a_1} x^{a_2} x^{c_1} \nabla^{\alpha} x^{d_1} \nabla_{\beta} x^{b_2} \nabla_{\gamma} x^{d_2} \,,
\end{aligned}
\tag{6.1}
$$

where $\bar{g} := g^{ab} - u^a u^b$. In all other cases we have

$$
\begin{aligned}
\bar{N}^{\alpha}{}_{\beta\gamma} &= \left(g^{b_2 d_1} - x^{b_2} x^{d_1}\right) S_{a_1 a_2 b_1 b_2} S_{c_1 c_2 d_1 d_2} x^{a_1} x^{a_2} x^{c_1} \nabla^{\alpha} x^{b_1} \nabla_{\beta} x^{d_2} \nabla_{\gamma} x^{c_2} \\
&+ \left(g^{b_1 c_2} - x^{b_1} x^{c_2}\right) S_{a_1 a_2 b_1 b_2} S_{c_1 c_2 d_1 d_2} x^{a_1} x^{a_2} x^{c_1} \nabla^{\alpha} x^{d_1} \nabla_{\beta} x^{b_2} \nabla_{\gamma} x^{d_2} \,.
\end{aligned}
$$

But here the two subtracted terms vanish by the Bianchi identity because they contain the terms $x^{a_1} x^{a_2} x^{b_2} S_{a_1 a_2 b_1 b_2}$ respectively $x^{a_1} x^{a_2} x^{b_1} S_{a_1 a_2 b_1 b_2}$. This allows us to use (6.1) for *all* models M if we define

$$
\bar{g}^{ab} :=
\begin{cases}
g^{ab} - u^a u^b & \text{if } M \text{ is flat} \\
g^{ab} & \text{otherwise} \,.
\end{cases}
\tag{6.2}
$$

In the case of a hyperplane $M \subset V$, the tensor $\bar{g}^{ab}$ is the pullback of the metric on M via the orthogonal projection $V \to M$ and thus degenerated. Note that we still lower and rise indices with the metric g^{ab} and not with $\bar{g}^{ab}$.

In (6.1) the lower indices b_2, d_1 respectively b_1, c_2 are contracted with $\bar{g}$. We can make use of the symmetries of $S_{a_1 a_2 b_1 b_2}$ to bring these indices to the first position:

$$
\begin{aligned}
\bar{N}^{\alpha}{}_{\beta\gamma} &= \bar{g}^{b_2 d_1} S_{b_2 b_1 a_1 a_2} S_{d_1 d_2 c_1 c_2} x^{a_1} x^{a_2} x^{c_1} \nabla^{\alpha} x^{b_1} \nabla_{\beta} x^{d_2} \nabla_{\gamma} x^{c_2} \\
&+ \bar{g}^{b_1 c_2} S_{b_1 b_2 a_1 a_2} S_{c_2 c_1 d_1 d_2} x^{a_1} x^{a_2} x^{c_1} \nabla^{\alpha} x^{d_1} \nabla_{\beta} x^{b_2} \nabla_{\gamma} x^{d_2} \,.
\end{aligned}
$$

Renaming, lowering and rising indices appropriately finally results in

$$
\bar{N}_{\alpha\beta\gamma} = \bar{g}_{ij}\left(S^i{}_{a_2 b_1 b_2} S^j{}_{c_2 d_1 d_2} + S^i{}_{c_2 b_1 b_2} S^j{}_{d_1 a_2 d_2}\right) x^{b_1} x^{b_2} x^{d_1} \nabla_{\alpha} x^{a_2} \nabla_{\beta} x^{c_2} \nabla_{\gamma} x^{d_2} \,.
\tag{6.3}
$$

In what follows we will substitute this expression together with (5.5a) into each of the three integrability conditions (2.3) and transform them into purely algebraic integrability conditions.

6.1. **The first integrability condition.** The first integrability condition (2.3a) can be written as $\bar{N}_{[\alpha\beta\gamma]} = 0$. For the expression (6.3) this is equivalent to the vanishing of the antisymmetrisation in the upper indices a_2, c_2, d_2:

$$
\bar{g}_{ij}\left(S^i{}_{a_2 b_1 b_2} S^j{}_{c_2 d_1 d_2} + S^i{}_{c_2 b_1 b_2} S^j{}_{d_1 a_2 d_2}\right) x^{b_1} x^{b_2} x^{d_1} \nabla_{\alpha} x^{[a_2} \nabla_{\beta} x^{c_2} \nabla_{\gamma} x^{d_2]} = 0 \,.
$$

Due to the symmetry (4.12a) the second term to vanishes. If we write u, v and w for the tangent vectors ∂_{α}, ∂_{β} respectively ∂_{γ} and use (5.3) in order to get rid of the indices and ∇'s, we get the condition

$$
\begin{aligned}
\bar{g}_{ij} S^i{}_{a_2 b_1 b_2} S^j{}_{c_2 d_1 d_2} x^{b_1} x^{b_2} x^{d_1} u^{[a_2} v^{c_2} w^{d_2]} &= 0 \\
\forall x \in M \quad \forall u, v, w \in T_x M
\end{aligned}
\tag{6.4}
$$

on the symmetrised algebraic curvature tensor $S_{a_1 a_2 b_1 b_2}$.

Note that tensors of the form $x^{b_1} x^{b_2} x^{d_1} u^{[a_2} v^{c_2} w^{d_2]}$ are completely symmetric in the indices b_1, b_2, d_1 and completely antisymmetric in the remaining indices

a_2, c_2, d_2. On the level of isomorphism classes the decomposition of the corresponding $\mathrm{GL}(V)$-representation space $\mathrm{Sym}^3 V \otimes \Lambda^3 V$ into irreducible components results from (4.8). The following lemma gives an explicit realisation of this decomposition in terms of orthogonal projection operators:

Lemma 6.2.

$$\frac{1}{q!}\,\begin{array}{|c|}\hline a_1 \\\hline \vdots \\\hline a_q \\\hline\end{array} \cdot \frac{1}{p!}\,\begin{array}{|c|c|c|}\hline s_1 & \cdots & s_p \\\hline\end{array} \tag{6.5}$$

$$= \frac{\frac{p}{q+1}}{(p+q)p!^2 q!^2}\,\begin{array}{|c|c|c|}\hline s_1 & \cdots & s_p \\\hline a_1 \\\cline{1-1}\vdots \\\cline{1-1} a_q \\\cline{1-1}\end{array}\begin{array}{|c|c|c|}\hline s_1 & \cdots & s_p \\\hline a_1 \\\cline{1-1}\vdots \\\cline{1-1} a_q \\\cline{1-1}\end{array}^{\!\star} + \frac{\frac{q}{p+1}}{(p+q)p!^2 q!^2}\,\begin{array}{|c|c|c|c|}\hline a_1 & s_1 & \cdots & s_p \\\hline \vdots \\\cline{1-1} a_q \\\cline{1-1}\end{array}^{\!\star}\begin{array}{|c|c|c|c|}\hline a_1 & s_1 & \cdots & s_p \\\hline \vdots \\\cline{1-1} a_q \\\cline{1-1}\end{array}$$

In particular, for $p = q = 3$:

$$\frac{1}{3!}\,\begin{array}{|c|}\hline c_2 \\\hline d_2 \\\hline a_2 \\\hline\end{array} \cdot \frac{1}{3!}\,\begin{array}{|c|c|c|}\hline b_2 & b_1 & d_1 \\\hline\end{array} = \frac{1}{2^7 3^4}\,\begin{array}{|c|c|c|}\hline b_2 & b_1 & d_1 \\\hline c_2 \\\cline{1-1} d_2 \\\cline{1-1} a_2 \\\cline{1-1}\end{array}\begin{array}{|c|c|c|}\hline b_2 & b_1 & d_1 \\\hline c_2 \\\cline{1-1} d_2 \\\cline{1-1} a_2 \\\cline{1-1}\end{array}^{\!\star} + \frac{1}{2^7 3^4}\,\begin{array}{|c|c|c|c|}\hline c_2 & b_2 & b_1 & d_1 \\\hline d_2 \\\cline{1-1} a_2 \\\cline{1-1}\end{array}^{\!\star}\begin{array}{|c|c|c|c|}\hline c_2 & b_2 & b_1 & d_1 \\\hline d_2 \\\cline{1-1} a_2 \\\cline{1-1}\end{array}. \tag{6.6}$$

Proof. Write (6.5) as $P = P_1 + P_2$. Decomposing temporarily the Young symmetrisers on the right hand side as in (4.1) into a product of a symmetriser and an antisymmetriser and using (4.3), one easily checks that P, P_1 and P_2 are orthogonal projectors verifying $P_1 P_2 = 0 = P_2 P_1$, $P P_1 = P_1$ and $P P_2 = P_2$. Therefore $P_1 + P_2$ is an orthogonal projector with image $\operatorname{im} P_1 \oplus \operatorname{im} P_2 \subseteq \operatorname{im} P$. The decomposition of the isomorphism class of $\operatorname{im} P$ into irreducible components is given by the Littlewood-Richardson rule as

$$q\left\{\;\begin{array}{|c|}\hline\;\\\hline\vdots\\\hline\;\\\hline\end{array}\;\boxtimes\;\overbrace{\begin{array}{|c|c|c|}\hline\;&\cdots&\;\\\hline\end{array}}^{p}\;\cong\;\overbrace{\begin{array}{|c|c|c|}\hline\;&\cdots&\;\\\hline\vdots\\\cline{1-1}\;\\\cline{1-1}\end{array}}^{p}\;\oplus\;\begin{array}{|c|c|c|c|}\hline\;&\;&\cdots&\;\\\hline\vdots\\\cline{1-1}\;\\\cline{1-1}\end{array}\;\right\}q\,.$$

The Young frames on the right hand side are those appearing in the expression for P_1 respectively P_2. Hence they describe the isomorphism classes of $\operatorname{im} P_1$ and $\operatorname{im} P_2$. This shows that $\operatorname{im} P$ and $\operatorname{im}(P_1 + P_2) = \operatorname{im} P_1 \oplus \operatorname{im} P_2$ have the same dimension and are thus equal. This implies $P = P_1 + P_2$. $\qquad\square$

Remark 6.3. *The lemma can be interpreted as an explicit splitting of the terms in the long exact sequence*

$$0 \to \mathrm{Sym}^d V \to \ldots \to \mathrm{Sym}^p V \otimes \Lambda^q V \to \mathrm{Sym}^{p-1}\otimes\Lambda^{q+1} \to \ldots \to \Lambda^d V \to 0\,,$$

known as the Koszul complex.

Applying (6.6) to the tensor $x^{b_1} x^{b_2} x^{d_1} u^{[a_2} v^{c_2} w^{d_2]}$, we conclude that

$$x^{b_1} x^{b_2} x^{d_1} u^{[a_2} v^{c_2} w^{d_2]}$$

$$= \frac{1}{2^7 3^4}\left(\begin{array}{|c|c|c|}\hline b_2 & b_1 & d_1 \\\hline\end{array}\,\begin{array}{|c|}\hline b_2 \\\hline c_2 \\\hline d_2 \\\hline a_2 \\\hline\end{array}^{\!2}\begin{array}{|c|c|c|}\hline b_2 & b_1 & d_1 \\\hline\end{array} + \begin{array}{|c|}\hline c_2 \\\hline d_2 \\\hline a_2 \\\hline\end{array}\begin{array}{|c|c|c|c|}\hline c_2 & b_2 & b_1 & d_1 \\\hline\end{array}^{\!2}\begin{array}{|c|}\hline c_2 \\\hline d_2 \\\hline a_2 \\\hline\end{array}\right) x^{b_1} x^{b_2} x^{d_1} u^{[a_2} v^{c_2} w^{d_2]}$$

$$= \mathrm{constant} \cdot \left(\begin{array}{|c|c|c|}\hline b_2 & b_1 & d_1 \\\hline c_2 \\\cline{1-1} d_2 \\\cline{1-1} a_2 \\\cline{1-1}\end{array} + \begin{array}{|c|c|c|c|}\hline c_2 & b_2 & b_1 & d_1 \\\hline d_2 \\\cline{1-1} a_2 \\\cline{1-1}\end{array}^{\!\star}\right) x^{b_1} x^{b_2} x^{d_1} u^{a_2} v^{c_2} w^{d_2}\,.$$

In the last step we omitted an explicit antisymmetrisation in a_2, c_2, d_2, since this is already carried out implicitly by each of the Young symmetrisers. Accordingly the left hand side of (6.4) splits into two terms. The following lemma shows that the second of them, namely

$$
\bar{g}_{ij} S^i{}_{a_2 b_1 b_2} S^j{}_{c_2 d_1 d_2}
\left(
\young(c_2b_2b_1d_1,d_2,a_2)^{\star}
\, x^{b_1} x^{b_2} x^{d_1} u^{a_2} v^{c_2} w^{d_2}
\right)
$$
$$
=
\left(
\young(c_2b_2b_1d_1,d_2,a_2)\,
\bar{g}_{ij} S^i{}_{a_2 b_1 b_2} S^j{}_{c_2 d_1 d_2}
\right)
x^{b_1} x^{b_2} x^{d_1} u^{a_2} v^{c_2} w^{d_2} \,,
$$

vanishes identically.

Lemma 6.4.

$$
\young(c_2b_2b_1d_1,d_2,a_2)\,
\bar{g}_{ij} S^i{}_{a_2 b_1 b_2} S^j{}_{c_2 d_1 d_2} = 0
\tag{6.7}
$$

Before we prove the lemma, we mention an identity which we will frequently use and which is obtained from symmetrising the Bianchi identity (4.12c) in b_1, b_2:

$$
\young(b_1b_2)\, S^i{}_{a_2 b_1 b_2} = -2\, \young(b_1b_2)\, S^i{}_{b_1 b_2 a_2} \,.
\tag{6.8}
$$

We refer to this identity as *symmetrised Bianchi identity*.

Proof.

$$
\young(c_2b_2b_1d_1,d_2,a_2)\,
\bar{g}_{ij} S^i{}_{a_2 b_1 b_2} S^j{}_{c_2 d_1 d_2}
=
\young(c_2b_2b_1d_1)\, \young(c_2,d_2,a_2)\,
\bar{g}_{ij} S^i{}_{a_2 b_1 b_2} S^j{}_{c_2 d_1 d_2}
$$

$$
=
\young(c_2b_2b_1d_1)\, g_{ij} \Big(\ \ S^i{}_{a_2 b_1 b_2} S^j{}_{c_2 d_1 d_2} - S^i{}_{d_2 b_1 b_2} S^j{}_{c_2 d_1 a_2}
$$
$$
+ S^i{}_{d_2 b_1 b_2} S^j{}_{a_2 d_1 c_2} - S^i{}_{a_2 b_1 b_2} S^j{}_{d_2 d_1 c_2}
$$
$$
+ S^i{}_{c_2 b_1 b_2} S^j{}_{d_2 d_1 a_2} - S^i{}_{c_2 b_1 b_2} S^j{}_{a_2 d_1 d_2} \Big)
$$

Regard the parenthesis under complete symmetrisation in c_2, b_2, b_1, d_1. The last two terms vanish due to the Bianchi identity (4.13). Renaming i, j as j, i in the third term shows that it cancels the fourth. That the first two also cancel each other can be seen by applying twice the symmetrised Bianchi identity (6.8), once to $S^i{}_{a_2 b_1 b_2}$ and once to $S^i{}_{c_2 d_1 d_2}$.   □

Remark 6.5. *If the inner product g is positive definite, then M is the unit sphere. In this case the lemma above also follows from the symmetry classification of Riemann tensor polynomials* [FKWC92], *since the tensor* $g_{ij} R^i{}_{a_2 b_1 b_2} R^j{}_{c_2 d_1 d_2}$ *has symmetry type*
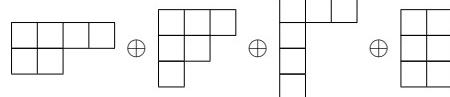
and $g_{ij} S^i{}_{a_2 b_1 b_2} S^j{}_{c_2 d_1 d_2}$ *can be expressed in terms of this tensor via* (4.14a).

Resuming, the first integrability condition is equivalent to

$$
\bar{g}_{ij} S^i{}_{a_2 b_1 b_2} S^j{}_{c_2 d_1 d_2}
\left(
\young(b_2b_1d_1,c_2,d_2,a_2)\,
x^{b_1} x^{b_2} x^{d_1} u^{a_2} v^{c_2} w^{d_2}
\right) = 0
\tag{6.9}
$$
$$
\forall x \in M \quad \forall u, v, w \in T_x M \,.
$$

We can drop the restriction $u, v, w \in T_x M$ in (6.9). Indeed, decomposing $u, v, w \in V$ according to the splitting $V = T_x M \oplus \mathbb{R} x$ shows

$$\young(b_2 b_1 d_1, c_2, d_2, a_2)\; x^{b_1} x^{b_2} x^{d_1} u^{a_2} v^{c_2} w^{d_2} = 0 \qquad \text{if } u = x \text{ or } v = x \text{ or } w = x.$$

This follows from Dirichlet's drawer principle. This trick is crucial, as it allows us to deal with $\mathrm{GL}(N)$-representations instead of the more complicated $\mathrm{O}(N)$-representations. Obviously, we can also drop the restriction $x \in M$ since $\mathbb{R}M$ is dense in V.

Next we use the fact that tensors of the form $x^{b_1} x^{b_2} x^{d_1}$ and tensors of the form $\young(b_2 b_1 d_1) x^{b_1} y^{b_2} z^{d_1}$ both span the same space, namely $\mathrm{Sym}^3 V$. With this remark condition (6.9) is now equivalent to

$$\bar{g}_{ij} S^i{}_{a_2 b_1 b_2} S^j{}_{c_2 d_1 d_2} \young(b_2 b_1 d_1, c_2, d_2, a_2)\; \young(b_2 b_1 d_1)\; x^{b_1} y^{b_2} z^{d_1} u^{a_2} v^{c_2} w^{d_2} = 0$$

$$\forall x, y, z, u, v, w \in V.$$

But the operator

$$\young(b_2 b_1 d_1, c_2, d_2, a_2)\; \young(b_2 b_1 d_1) = \young(b_2 b_1 d_1)\; \young(b_2, c_2, d_2, a_2)\; \young(b_2 b_1 d_1) = \frac{1}{4!}\, \young(b_2 b_1 d_1)\; \young(b_2, c_2, d_2, a_2)\; \young(b_2, c_2, d_2, a_2)\; \young(b_2 b_1 d_1)$$

$$= \frac{1}{4!}\, \young(b_2 b_1 d_1, c_2, d_2, a_2)\; \young(b_2 b_1 d_1, c_2, d_2, a_2)^{\star}$$

is self-adjoint and hence

$$\bar{g}_{ij} S^i{}_{a_2 b_1 b_2} S^j{}_{c_2 d_1 d_2} \left(\young(b_2 b_1 d_1, c_2, d_2, a_2)\; \young(b_2 b_1 d_1)\; x^{b_1} y^{b_2} z^{d_1} u^{a_2} v^{c_2} w^{d_2} \right)$$

$$= \frac{1}{4!} \left(\young(b_2 b_1 d_1, c_2, d_2, a_2)\; \young(b_2 b_1 d_1, c_2, d_2, a_2)^{\star} \bar{g}_{ij} S^i{}_{a_2 b_1 b_2} S^j{}_{c_2 d_1 d_2} \right) x^{b_1} y^{b_2} z^{d_1} u^{a_2} v^{c_2} w^{d_2}.$$

Now recall that $V^{\otimes 6}$ is spanned by tensors of the form $x^{b_1} y^{b_2} z^{d_1} u^{a_2} v^{c_2} w^{d_2}$ and that the dual pairing $\bar{V}^{\otimes 6} \times V^{\otimes 6} \to \mathbb{R}$ is non-degenerate. This allows us finally to write the first integrability condition in the purely algebraic form

$$\young(b_2 b_1 d_1, c_2, d_2, a_2)\; \young(b_2 b_1 d_1, c_2, d_2, a_2)^{\star} \bar{g}_{ij} S^i{}_{a_2 b_1 b_2} S^j{}_{c_2 d_1 d_2} = 0, \tag{6.10}$$

which is independent of x, y, z, u, v, w. We will now give a number of equivalent formulations.

Proposition 6.6 (First integrability condition)**.** *The following conditions are equivalent to the first integrability condition* (2.3a) *for a Killing tensor on a constant sectional curvature manifold M:*

(i) *The corresponding symmetrised algebraic curvature tensor S satisfies*

$$P\,\bar{g}_{ij}S^i{}_{a_2b_1b_2}S^j{}_{c_2d_1d_2}=0\,,\tag{6.11}$$

where P is any of the following symmetry operators [5]

$$(a)\;\young(b_2b_1d_1,c_2,d_2,a_2)^{\star}\qquad(b)\;\young(c_2,d_2,a_2)\;\young(b_2b_1d_1)\qquad(c)\;\young(b_2,c_2,d_2,a_2)\qquad(d)\;\young(b_2b_1d_1,c_2,d_2,a_2)\tag{6.12}$$

(ii) *The corresponding algebraic curvature tensor R satisfies*

$$P\,\bar{g}_{ij}R^i{}_{b_1a_2b_2}R^j{}_{d_1c_2d_2}=0\,,\tag{6.13}$$

where P is any of the symmetry operators (6.12).

If M is not flat, this is is in addition equivalent to:

(iii) *The curvature form $\Omega\in\operatorname{End}(V)\otimes\Lambda^2V$ of R, defined by*

$$\Omega^{a_1}{}_{b_1}:=R^{a_1}{}_{b_1a_2b_2}dx^{a_2}\wedge dx^{b_2}$$

satisfies

$$\Omega\wedge\Omega=0\,,\tag{6.14}$$

where the wedge product is defined by taking the exterior product in the Λ^2V-component and usual matrix multiplication in the $\operatorname{End}(V)$-component.

Remark 6.7. *In* (6.12) *we can permute the labels a_2,b_2,c_2,d_2 arbitrarily as well as exchange the labels b_1,d_1. This follows from the integrability condition in the form* (6.11c). *In particular, in* (6.11b) *one can antisymmetrise in any three of the four indices a_2,b_2,c_2,d_2 and symmetrise in the remaining three.*

Proof. We showed that the first integrability condition (2.3a) is equivalent to (6.10). But this is equivalent to condition (6.11a) since the kernels of $PP^\star$ and $P^\star$ coincide:

$$PP^\star v=0\quad\Leftrightarrow\quad\|P^\star v\|^2=\langle v|PP^\star v\rangle=0\quad\Leftrightarrow\quad P^\star v=0\,.\tag{6.15}$$

The equivalence (6.11a) $\Leftrightarrow$ (6.11b) follows from (6.6) combined with (6.7).

The implication (6.11c) $\Rightarrow$ (6.11d) is trivial. We finish the proof of part (i) by proving (6.11d) $\Rightarrow$ (6.11b) $\Rightarrow$ (6.11c) through a stepwise manipulation of

$$\young(b_2b_1d_1,c_2,d_2,a_2)\;\bar{g}_{ij}S^i{}_{a_2b_1b_2}S^j{}_{c_2d_1d_2}=\young(b_2b_1d_1)\;\young(b_2,c_2,d_2,a_2)\;\bar{g}_{ij}S^i{}_{a_2b_1b_2}S^j{}_{c_2d_1d_2}\,.\tag{6.16a}$$

In order to sum over all $q!$ permutations of q indices, one can take the sum over q cyclic permutations, chose one index and then sum over all $(q-1)!$ permutations of the remaining $(q-1)$ indices. Apply this to the antisymmetrisation in a_2,b_2,c_2,d_2 (fixing b_2):

$$(6.16a)=\young(b_2b_1d_1)\;\young(c_2,d_2,a_2)\;\bar{g}_{ij}\Big(S^i{}_{\underline{a}_2b_1b_2}S^j{}_{\underline{c}_2d_1\underline{d}_2}-S^i{}_{b_2b_1\underline{c}_2}S^j{}_{\underline{d}_2d_1\underline{a}_2}$$

$$+S^i{}_{\underline{c}_2b_1\underline{d}_2}S^j{}_{\underline{a}_2d_1b_2}-S^i{}_{\underline{d}_2b_1\underline{a}_2}S^j{}_{b_2d_1\underline{c}_2}\Big)\,.\tag{6.16b}$$

---

[5]Up to a constant they are all projectors.

For a better readability we underlined each antisymmetrised index. Now use the symmetrised Bianchi identity (6.8) to bring the index c_2 from the fourth to the second index position:

$$(6.16\text{b}) = \boxed{b_2\,b_1\,d_1}\,\boxed{\begin{smallmatrix}c_2\\d_2\\a_2\end{smallmatrix}}\,\bar{g}_{ij}\Big(\ S^i{}_{\underline{a}_2 b_1 b_2}S^j{}_{\underline{c}_2 d_1 \underline{d}_2} + \tfrac{1}{2}S^i{}_{\underline{c}_2 b_1 b_2}S^j{}_{\underline{d}_2 d_1 \underline{a}_2}$$
$$+ S^i{}_{\underline{c}_2 b_1 \underline{d}_2}S^j{}_{\underline{a}_2 d_1 b_2} + \tfrac{1}{2}S^i{}_{\underline{d}_2 b_1 \underline{a}_2}S^j{}_{\underline{c}_2 d_1 b_2}\Big). \tag{6.16c}$$

Then rename i, j as j, i in the last two terms:

$$(6.16\text{c}) = \boxed{b_2\,b_1\,d_1}\,\boxed{\begin{smallmatrix}c_2\\d_2\\a_2\end{smallmatrix}}\,\bar{g}_{ij}\Big(\ S^i{}_{\underline{a}_2 b_1 b_2}S^j{}_{\underline{c}_2 d_1 \underline{d}_2} + \tfrac{1}{2}S^i{}_{\underline{c}_2 b_1 b_2}S^j{}_{\underline{d}_2 d_1 \underline{a}_2}$$
$$+ S^i{}_{\underline{a}_2 d_1 b_2}S^j{}_{\underline{c}_2 b_1 \underline{d}_2} + \tfrac{1}{2}S^i{}_{\underline{c}_2 d_1 b_2}S^j{}_{\underline{d}_2 b_1 \underline{a}_2}\Big). \tag{6.16d}$$

Finally use the symmetrisation in b_2, b_1, d_1 and the antisymmetrisation in c_2, d_2, a_2 to bring each term to the same form:

$$(6.16\text{d}) = \boxed{b_2\,b_1\,d_1}\,\boxed{\begin{smallmatrix}c_2\\d_2\\a_2\end{smallmatrix}}\,\bar{g}_{ij}\Big(3S^i{}_{\underline{a}_2 b_1 b_2}S^j{}_{\underline{c}_2 d_1 \underline{d}_2}\Big). \tag{6.16e}$$

This proves (6.11d) $\Leftrightarrow$ (6.11b). To continue, antisymmetrise

$$0 = \boxed{\begin{smallmatrix}c_2\\d_2\\a_2\end{smallmatrix}}\,\boxed{b_2\,b_1\,d_1}\,\bar{g}_{ij}S^i{}_{\underline{a}_2 b_1 b_2}S^j{}_{\underline{c}_2 d_1 \underline{d}_2}$$
$$= 2\,\boxed{\begin{smallmatrix}c_2\\d_2\\a_2\end{smallmatrix}}\,\bar{g}_{ij}\Big(S^i{}_{\underline{a}_2 b_1 b_2}S^j{}_{\underline{c}_2 d_1 \underline{d}_2} + S^i{}_{\underline{a}_2 b_2 d_1}S^j{}_{\underline{c}_2 b_1 \underline{d}_2} + S^i{}_{\underline{a}_2 d_1 b_1}S^j{}_{\underline{c}_2 b_2 \underline{d}_2}\Big)$$

in a_2, b_2, c_2, d_2. Then the last term vanishes by the symmetry (4.12a), yielding

$$0 = \boxed{\begin{smallmatrix}a_2\\b_2\\c_2\\d_2\end{smallmatrix}}\,\bar{g}_{ij}\Big(S^i{}_{\underline{a}_2 b_1 \underline{b}_2}S^j{}_{\underline{c}_2 d_1 \underline{d}_2} + S^i{}_{\underline{a}_2 d_1 \underline{b}_2}S^j{}_{\underline{c}_2 b_1 \underline{d}_2}\Big).$$

Both sum terms are equal under antisymmetrisation in a_2, b_2, c_2, d_2 and contraction with $\bar{g}_{ij}$. Indeed, exchanging b_1 and d_1 is tantamount to exchanging a_2 with c_2 and b_2 with d_2 and renaming i, j as j, i. This proves (6.11b) $\Rightarrow$ (6.11c).

From the correspondence (4.14) between R and S we conclude the equivalence (6.11c) $\Leftrightarrow$ (6.13c). The proof of the remaining part of (ii) is completely analogous to the proof of (i), so we leave it to the reader. Condition (6.14) is just a reformulation of (6.13c). This finishes the proof. $\qquad\square$

Remark 6.8. *In the preceeding proof we made use of a particular notation as well as some particular tensor index manipulations. We will do this several times in more complex computations during the next two sections, so we would like to make this explicit.*

- *First, we call a Young symmetriser as in (6.16a), which is the product of a symmetriser and an antisymmetriser sharing a common label (and thus not commuting) a* hook symmetriser. *Note that (6.16a) and (6.16b) are merely different ways to write down the same term, using a smaller antisymmetriser but applied to more terms. We call this to* reduce an antisymmetriser by a label *(b_2 in this case). This works likewise for a symmetriser*

and allows us to replace any hook symmetriser by a product of a symmetriser and an antisymmetriser with disjoint label sets (and thus both commuting). The latter are more easy to deal with. We call this procedure splitting a hook symmetriser.

- *Second, for better readability we stick to the above notation and underline antisymmetrised tensor indices as in* (6.16c).
- *Third, regard the manipulations from* (6.16c) *to* (6.16e). *What we did is to bring the indices of every term in* (6.16c) *to the same order as in* $g_{ij} S^i{}_{a_2 b_1 b_2} S^j{}_{c_2 d_1 d_2}$ *by using:*
  - *the symmetry in* i, j *under contraction with* $\bar{g}_{ij}$,
  - *the (anti)symmetry under the (anti)symmetriser and*
  - *the symmetries of S itself, especially the symmetrised Bianchi identity* (6.8).

We will call this procedure reordering indices.

6.2. **The second integrability condition.** The proceeding for the remaining two integrability conditions is similar as for the first one, only longer. We therefore treat both in parallel as far as possible and shorten the explications where they are analogous. We begin by substituting the expressions (6.3) and (5.5a) into the tensors appearing in (2.3b) and (2.3c):

$$
\begin{aligned}
&\bar{N}^\delta{}_{\beta\gamma} K_{\delta\alpha} \\
&= \bar{g}_{ij} \big(S^i{}_{a_2 b_1 b_2} S^j{}_{c_2 d_1 d_2} + S^i{}_{c_2 b_1 b_2} S^j{}_{d_1 a_2 d_2} \big) x^{b_1} x^{b_2} x^{d_1} \nabla^\delta x^{a_2} \nabla_\beta x^{c_2} \nabla_\gamma x^{d_2} \\
&\qquad S_{e_1 e_2 f_1 f_2} x^{e_1} x^{e_2} \nabla_\delta x^{f_1} \nabla_\alpha x^{f_2} \\
&\bar{N}^\delta{}_{\beta\gamma} K_{\varepsilon\alpha} K^\varepsilon{}_\delta \\
&= \bar{g}_{ij} \big(S^i{}_{a_2 b_1 b_2} S^j{}_{c_2 d_1 d_2} + S^i{}_{c_2 b_1 b_2} S^j{}_{d_1 a_2 d_2} \big) x^{b_1} x^{b_2} x^{d_1} \nabla^\delta x^{a_2} \nabla_\beta x^{c_2} \nabla_\gamma x^{d_2} \\
&\qquad S_{e_1 e_2 f_1 f_2} x^{e_1} x^{e_2} \nabla_\varepsilon x^{f_1} \nabla_\alpha x^{f_2} \ S_{g_1 g_2 h_1 h_2} x^{g_1} x^{g_2} \nabla^\varepsilon x^{h_1} \nabla_\delta x^{h_2} \ .
\end{aligned}
$$

As before, we replace the contractions over δ and ε according to lemma 6.1 and omit the terms that vanish according to the Bianchi identity:

$$
\begin{aligned}
&\bar{N}^\delta{}_{\beta\gamma} K_{\delta\alpha} \\
&= \bar{g}_{ij} \bar{g}^{a_2 f_1} \big(S^i{}_{a_2 b_1 b_2} S^j{}_{c_2 d_1 d_2} + S^i{}_{c_2 b_1 b_2} S^j{}_{d_1 a_2 d_2} \big) S_{e_1 e_2 f_1 f_2} \\
&\qquad x^{b_1} x^{b_2} x^{d_1} x^{e_1} x^{e_2} \nabla_\beta x^{c_2} \nabla_\gamma x^{d_2} \nabla_\alpha x^{f_2} \\
&\bar{N}^\delta{}_{\beta\gamma} K_{\varepsilon\alpha} K^\varepsilon{}_\delta \\
&= \bar{g}_{ij} \bar{g}^{a_2 h_2} \bar{g}^{f_1 h_1} \big(S^i{}_{a_2 b_1 b_2} S^j{}_{c_2 d_1 d_2} + S^i{}_{c_2 b_1 b_2} S^j{}_{d_1 a_2 d_2} \big) S_{e_1 e_2 f_1 f_2} S_{g_1 g_2 h_1 h_2} \\
&\qquad x^{b_1} x^{b_2} x^{d_1} x^{e_1} x^{e_2} x^{g_1} x^{g_2} \nabla_\beta x^{c_2} \nabla_\gamma x^{d_2} \nabla_\alpha x^{f_2} \ .
\end{aligned}
$$

The integrability conditions (2.3b) and (2.3c) are equivalent to the vanishing of the antisymmetrisation of the above tensors in α, β, γ. As before, this can be written

as

$$\bar{g}_{ij}\bar{g}_{kl}\big(S^{ik}{}_{b_1 b_2}S^{j}{}_{c_2 d_1 d_2} + S^{i}{}_{c_2 b_1 b_2}S^{j}{}_{d_1}{}^{k}{}_{d_2}\big)S^{l}{}_{f_2 e_1 e_2}$$
$$x^{b_1}x^{b_2}x^{d_1}x^{e_1}x^{e_2}u^{[c_2}v^{d_2}w^{f_2]} \;=\; 0$$

$$\bar{g}_{ij}\bar{g}_{kl}\bar{g}_{mn}\big(S^{ik}{}_{b_1 b_2}S^{j}{}_{c_2 d_1 d_2} + S^{i}{}_{c_2 b_1 b_2}S^{j}{}_{d_1}{}^{k}{}_{d_2}\big)S^{m}{}_{f_2 e_1 e_2}S^{nl}{}_{g_1 g_2} \tag{6.17}$$
$$x^{b_1}x^{b_2}x^{d_1}x^{e_1}x^{e_2}x^{g_1}x^{g_2}u^{[c_2}v^{d_2}w^{f_2]} \;=\; 0$$

$$\forall x \in M \qquad \forall u, v, w \in T_x M \,.$$

The tensors

$$x^{b_1}x^{b_2}x^{d_1}x^{e_1}x^{e_2}u^{[c_2}v^{d_2}w^{f_2]} \qquad\qquad x^{b_1}x^{b_2}x^{d_1}x^{e_1}x^{e_2}x^{g_1}x^{g_2}u^{[c_2}v^{d_2}w^{f_2]}$$

are antisymmetric in c_2, d_2, f_2 and symmetric in the remaining indices. We decompose them according to lemma 6.2. This yields

$$x^{b_1}x^{b_2}x^{d_1}x^{e_1}x^{e_2}u^{[c_2}v^{d_2}w^{f_2]}$$

$$= \text{constant} \cdot \young(b_2 b_1 d_1 e_1 e_2,c_2,d_2,f_2) \quad x^{b_1}x^{b_2}x^{d_1}x^{e_1}x^{e_2}u^{c_2}v^{d_2}w^{f_2}$$

$$+ \text{constant} \cdot \young(c_2 b_2 b_1 d_1 e_1 e_2,d_2,f_2)^{\star} \quad x^{b_1}x^{b_2}x^{d_1}x^{e_1}x^{e_2}u^{c_2}v^{d_2}w^{f_2}$$

and

$$x^{b_1}x^{b_2}x^{d_1}x^{e_1}x^{e_2}x^{g_1}x^{g_2}u^{[c_2}v^{d_2}w^{f_2]}$$

$$= \text{constant} \cdot \young(b_2 b_1 d_1 e_1 e_2 g_1 g_2,c_2,d_2,f_2) \quad x^{b_1}x^{b_2}x^{d_1}x^{e_1}x^{e_2}x^{g_1}x^{g_2}u^{c_2}v^{d_2}w^{f_2}$$

$$+ \text{constant} \cdot \young(c_2 b_2 b_1 d_1 e_1 e_2 g_1 g_2,d_2,f_2)^{\star} \quad x^{b_1}x^{b_2}x^{d_1}x^{e_1}x^{e_2}x^{g_1}x^{g_2}u^{c_2}v^{d_2}w^{f_2} \,.$$

The following lemma shows that, when substituted into (6.17), only the first term is relevant in each case:

Lemma 6.9.

$$\young(c_2 b_2 b_1 d_1 e_1 e_2,d_2,f_2) \quad \bar{g}_{ij}\bar{g}_{kl}\, S^{ik}{}_{b_1 b_2}S^{j}{}_{c_2 d_1 d_2}S^{l}{}_{f_2 e_1 e_2} = 0 \tag{6.18a}$$

$$\young(c_2 b_2 b_1 d_1 e_1 e_2,d_2,f_2) \quad \bar{g}_{ij}\bar{g}_{kl}S^{i}{}_{c_2 b_1 b_2}S^{j}{}_{d_1}{}^{k}{}_{d_2}\, S^{l}{}_{f_2 e_1 e_2} = 0 \tag{6.18b}$$

$$\young(c_2 b_2 b_1 d_1 e_1 e_2 g_1 g_2,d_2,f_2) \quad \bar{g}_{ij}\bar{g}_{kl}\bar{g}_{mn}\, S^{ik}{}_{b_1 b_2}S^{j}{}_{c_2 d_1 d_2}S^{m}{}_{f_2 e_1 e_2}S^{nl}{}_{g_1 g_2} = 0 \tag{6.19a}$$

$$\young(c_2 b_2 b_1 d_1 e_1 e_2 g_1 g_2,d_2,f_2) \quad \bar{g}_{ij}\bar{g}_{kl}\bar{g}_{mn}S^{i}{}_{c_2 b_1 b_2}S^{j}{}_{d_1}{}^{k}{}_{d_2}\, S^{m}{}_{f_2 e_1 e_2}S^{nl}{}_{g_1 g_2} = 0 \tag{6.19b}$$

Proof. Expanding the antisymmetriser of the Young symmetriser on the left hand side of (6.18a) yields

$$\boxed{c_2 \,|\, b_2 \,|\, b_1 \,|\, d_1 \,|\, e_1 \,|\, e_2}\, \bar{g}_{ij}\bar{g}_{kl}S^{ik}{}_{b_1b_2}\big(\quad S^{j}{}_{c_2d_1d_2}S^{l}{}_{f_2e_1e_2} - S^{j}{}_{c_2d_1f_2}S^{l}{}_{d_2e_1e_2}$$
$$+ S^{j}{}_{f_2d_1c_2}S^{l}{}_{d_2e_1e_2} - S^{j}{}_{d_2d_1c_2}S^{l}{}_{f_2e_1e_2}$$
$$+ S^{j}{}_{d_2d_1f_2}S^{l}{}_{c_2e_1e_2} - S^{j}{}_{f_2d_1d_2}S^{l}{}_{c_2e_1e_2}\big)\,.$$

Now regard the parenthesis under complete symmetrisation in $b_1, b_2, c_2, d_1, e_1, e_2$. The last two terms vanish by the Bianchi identity. Renaming i, j, k, l as k, l, i, j in the third term shows that it cancels the fourth due to the contraction with $\bar{g}_{ij}\bar{g}_{kl}S^{ik}{}_{b_1b_2}$. That the first two also cancel each other can be seen after applying twice the symmetrised Bianchi identity, once to $S^{j}{}_{c_2d_1d_2}$ and once to $S^{l}{}_{f_2e_1e_2}$.

In the same way, the left hand side of (6.18b), written without terms vanishing by the Bianchi identity, is

$$\boxed{c_2 \,|\, b_2 \,|\, b_1 \,|\, d_1 \,|\, e_1 \,|\, e_2}\, \bar{g}_{ij}\bar{g}_{kl}S^{j}{}_{d_1}{}^{k}{}_{c_2}\big(S^{i}{}_{d_2b_1b_2}S^{l}{}_{f_2e_1e_2} - S^{i}{}_{f_2b_1b_2}S^{l}{}_{d_2e_1e_2}\big)\,.$$

Renaming i, j, k, l as l, k, j, i in the first term shows that this is zero too. The proof of (6.19) is straightforward, using the same arguments. We leave this to the reader. □

Remark 6.10. *For the unit sphere, the lemma also follows from the symmetry classification of Riemann tensor polynomials* [FKWC92]. *Indeed, the tensors under the Young symmetriser in* (6.18) *and* (6.19) *can be expressed in terms of the corresponding algebraic curvature tensor via* (4.14a) *and the resulting tensors have no ⬚⬚⬚⬚ respectively ⬚⬚⬚⬚⬚ component.*

We have shown the equivalence of the second and third integrability condition to

$$\bar{g}_{ij}\bar{g}_{kl}\big(S^{ik}{}_{b_1b_2}S^{j}{}_{c_2d_1d_2} + S^{i}{}_{c_2b_1b_2}S^{j}{}_{d_1}{}^{k}{}_{d_2}\big)S^{l}{}_{f_2e_1e_2}$$
$$\boxed{\begin{array}{|c|c|c|c|c|} b_2 & b_1 & d_1 & e_1 & e_2 \\ \hline c_2 \\ \hline d_2 \\ \hline f_2 \end{array}}\; x^{b_1}x^{b_2}x^{d_1}x^{e_1}x^{e_2}u^{c_2}v^{d_2}w^{f_2} \;=\; 0$$

$$\bar{g}_{ij}\bar{g}_{kl}\bar{g}_{mn}\big(S^{ik}{}_{b_1b_2}S^{j}{}_{c_2d_1d_2} + S^{i}{}_{c_2b_1b_2}S^{j}{}_{d_1}{}^{k}{}_{d_2}\big)S^{m}{}_{f_2e_1e_2}S^{nl}{}_{g_1g_2}$$
$$\boxed{\begin{array}{|c|c|c|c|c|c|c|} b_2 & b_1 & d_1 & e_1 & e_2 & g_1 & g_2 \\ \hline c_2 \\ \hline d_2 \\ \hline f_2 \end{array}}\; x^{b_1}x^{b_2}x^{d_1}x^{e_1}x^{e_2}x^{g_1}x^{g_2}u^{c_2}v^{d_2}w^{f_2} \;=\; 0$$

$$\forall x \in M \quad \forall u, v, w \in T_x M$$

respectively. As before, the restrictions $\forall u, v, w \in T_x M$ and $\forall x \in M$ can be dropped and this allows us to write both conditions independently of the vectors x, u, v, w

as

$$\boxed{\begin{array}{|c|c|c|c|c|}\hline b_2 & b_1 & d_1 & e_1 & e_2 \\\hline c_2 \\\cline{1-1} d_2 \\\cline{1-1} f_2 \\\cline{1-1}\end{array}}^{\star} \bar{g}_{ij}\bar{g}_{kl}\big(S^{ik}{}_{b_1 b_2}S^j{}_{c_2 d_1 d_2}+S^i{}_{c_2 b_1 b_2}S^j{}_{d_1}{}^k{}_{d_2}\big)S^l{}_{f_2 e_1 e_2}=0 \qquad (6.20)$$

$$\boxed{\begin{array}{|c|c|c|c|c|c|c|}\hline b_2 & b_1 & d_1 & e_1 & e_2 & g_1 & g_2 \\\hline c_2 \\\cline{1-1} d_2 \\\cline{1-1} f_2 \\\cline{1-1}\end{array}}^{\star} \bar{g}_{ij}\bar{g}_{kl}\bar{g}_{mn}\big(S^{ik}{}_{b_1 b_2}S^j{}_{c_2 d_1 d_2}+S^i{}_{c_2 b_1 b_2}S^j{}_{d_1}{}^k{}_{d_2}\big)S^m{}_{f_2 e_1 e_2}S^{nl}{}_{g_1 g_2}=0\,.$$

$$(6.21)$$

In order to simplify these conditions we need the following two lemmas.

Lemma 6.11. *The first integrability condition is equivalent to*

$$\boxed{\begin{array}{|c|}\hline b_2 \\\hline c_2 \\\hline d_2 \\\hline\end{array}}\ \boxed{\begin{array}{|c|c|}\hline b_1 & d_1 \\\hline\end{array}}\ \bar{g}_{ij}\big(S^i{}_{b_1}{}^k{}_{\underline{b}_2}+2S^i{}_{\underline{b}_2}{}^k{}_{b_1}\big)S^j{}_{\underline{c}_2 d_1 \underline{d}_2}=0\,. \qquad (6.22)$$

Proof. Take the first integrability condition in the form (6.11c) and reduce the antisymmetriser by the label a_2:

$$\boxed{\begin{array}{|c|}\hline b_2 \\\hline c_2 \\\hline d_2 \\\hline\end{array}}\ \bar{g}_{ij}\big(\ \ S^i{}_{a_2 b_1 \underline{b}_2}S^j{}_{\underline{c}_2 d_1 \underline{d}_2}-S^i{}_{\underline{b}_2 b_1 \underline{c}_2}S^j{}_{\underline{d}_2 d_1 a_2}$$
$$+S^i{}_{\underline{c}_2 b_1 \underline{d}_2}S^j{}_{a_2 d_1 \underline{b}_2}-S^i{}_{\underline{d}_2 b_1 a_2}S^j{}_{\underline{b}_2 d_1 \underline{c}_2}\ \big)=0\,.$$

If we symmetrise this expression in b_1, d_1, the first and third as well as the second and fourth term become equal. Permuting indices, we get

$$\boxed{\begin{array}{|c|}\hline b_2 \\\hline c_2 \\\hline d_2 \\\hline\end{array}}\ \boxed{\begin{array}{|c|c|}\hline b_1 & d_1 \\\hline\end{array}}\ \bar{g}_{ij}\big(S^i{}_{a_2 b_1 \underline{b}_2}-S^i{}_{\underline{b}_2 a_2 b_1}\big)S^j{}_{\underline{c}_2 d_1 \underline{d}_2}=0\,. \qquad (6.23)$$

If we now symmetrise in a_2, b_1, d_1 and apply the symmetrised Bianchi identity to $S^i{}_{a_2 b_1 \underline{b}_2}$, we get back the first integrability condition in the form (6.11b). This proves its equivalence to (6.23). Applying now the Bianchi identity to the first term in (6.23) yields (6.22) with the index k lowered and renamed as a_2. □

Lemma 6.12. *The following identity is a consequence of the first integrability condition:*

$$\boxed{\begin{array}{|c|}\hline c_2 \\\hline d_2 \\\hline\end{array}}\ \boxed{\begin{array}{|c|c|c|}\hline b_2 & b_1 & d_1 \\\hline\end{array}}\ \bar{g}_{ij}\big(S^i{}_{b_1}{}^k{}_{b_2}S^j{}_{\underline{c}_2 d_1 \underline{d}_2}-\tfrac{1}{2}S^i{}_{d_1}{}^k{}_{\underline{d}_2}S^j{}_{\underline{c}_2 b_1 b_2}-S^i{}_{\underline{d}_2}{}^k{}_{d_1}S^j{}_{\underline{c}_2 b_1 b_2}\big)=0\,. \qquad (6.24)$$

Proof. Reduce the antisymmetriser in (6.22) by the index b_2,

$$\boxed{\begin{array}{|c|}\hline c_2 \\\hline d_2 \\\hline\end{array}}\ \boxed{\begin{array}{|c|c|}\hline b_1 & d_1 \\\hline\end{array}}\ \bar{g}_{ij}\big(\ \ S^i{}_{b_1}{}^k{}_{b_2}S^j{}_{\underline{c}_2 d_1 \underline{d}_2}+2S^i{}_{b_2}{}^k{}_{b_1}S^j{}_{\underline{c}_2 d_1 \underline{d}_2}$$
$$+S^i{}_{b_1}{}^k{}_{\underline{c}_2}S^j{}_{\underline{d}_2 d_1 b_2}+2S^i{}_{\underline{c}_2}{}^k{}_{b_1}S^j{}_{\underline{d}_2 d_1 b_2}$$
$$+S^i{}_{b_1}{}^k{}_{\underline{d}_2}S^j{}_{b_2 d_1 \underline{c}_2}+2S^i{}_{\underline{d}_2}{}^k{}_{b_1}S^j{}_{b_2 d_1 \underline{c}_2}\ \big)=0\,,$$

and then symmetrise in b_2, b_1, d_1. In the last line we can then apply the symmetrised Bianchi identity in order to move the antisymmetrised index c_2 from the fourth to

the second position:

$$\boxed{\begin{smallmatrix}c_2\\d_2\end{smallmatrix}}\;\boxed{b_2|b_1|\underline{d}_1}\,\bar g_{ij}\big(\quad S^i{}_{b_1}{}^k{}_{b_2}S^j{}_{\underline{c}_2 d_1\underline{d}_2}+2S^i{}_{b_2}{}^k{}_{b_1}S^j{}_{\underline{c}_2 d_1\underline{d}_2}$$
$$-\;\;S^i{}_{b_1}{}^k{}_{\underline{d}_2}S^j{}_{\underline{c}_2 d_1 b_2}-2S^i{}_{\underline{d}_2}{}^k{}_{b_1}S^j{}_{\underline{c}_2 d_1 b_2}$$
$$-\tfrac12 S^i{}_{b_1}{}^k{}_{\underline{d}_2}S^j{}_{\underline{c}_2 d_1 b_2}-\;\;S^i{}_{\underline{d}_2}{}^k{}_{b_1}S^j{}_{\underline{c}_2 d_1 b_2}\big)=0\,.$$

After permuting indices under symmetrisation appropriately, we get the desired
result. $\qquad\square$

Proposition 6.13 (Second integrability condition). *Suppose a Killing tensor on
a constant sectional curvature manifold satisfies the first integrability condition
(2.3a). Then the following conditions are equivalent to the second integrability con-
dition (2.3b):*

(i) *The corresponding symmetrised algebraic curvature tensor S satisfies one
of the following two equivalent conditions:*

$$\boxed{\begin{smallmatrix}b_2|b_1|d_1|e_1|e_2\\\hline c_2\\\hline d_2\\\hline f_2\end{smallmatrix}}^{\!\!\star}\;\bar g_{ij}\bar g_{kl}S^i{}_{c_2 d_1 d_2}S^j{}_{b_1}{}^k{}_{b_2}S^l{}_{f_2 e_1 e_2}=0 \tag{6.25a}$$

$$\boxed{\begin{smallmatrix}b_2|b_1|d_1|e_1|e_2\\\hline c_2\\\hline d_2\\\hline f_2\end{smallmatrix}}^{\!\!\star}\;\bar g_{ij}\bar g_{kl}\,S^i{}_{c_2 b_1 b_2}S^j{}_{d_1}{}^k{}_{d_2}S^l{}_{f_2 e_1 e_2}=0\,. \tag{6.25b}$$

(ii) *The corresponding symmetrised algebraic curvature tensor S satisfies one
of the following two equivalent conditions:*

$$\boxed{\begin{smallmatrix}c_2\\\hline d_2\\\hline f_2\end{smallmatrix}}\;\boxed{b_2|b_1|d_1|e_1|e_2}\,\bar g_{ij}\bar g_{kl}S^i{}_{\underline{c}_2 d_1\underline{d}_2}S^j{}_{b_1}{}^k{}_{b_2}S^l{}_{\underline{f}_2 e_1 e_2}=0 \tag{6.26a}$$

$$\boxed{\begin{smallmatrix}c_2\\\hline d_2\\\hline f_2\end{smallmatrix}}\;\boxed{b_2|b_1|d_1|e_1|e_2}\,\bar g_{ij}\bar g_{kl}S^i{}_{\underline{c}_2 b_1 b_2}S^j{}_{d_1}{}^k{}_{\underline{d}_2}S^l{}_{\underline{f}_2 e_1 e_2}=0\,. \tag{6.26b}$$

(iii) *The corresponding symmetrised algebraic curvature tensor S satisfies one
of the following three equivalent conditions:*

$$\boxed{\begin{smallmatrix}b_2\\\hline c_2\\\hline d_2\\\hline f_2\end{smallmatrix}}\;\boxed{b_1|d_1|e_1|e_2}\;\begin{cases}\bar g_{ij}\bar g_{kl}S^i{}_{\underline{c}_2 d_1\underline{d}_2}S^j{}_{b_1}{}^k{}_{\underline{b}_2}S^l{}_{\underline{f}_2 e_1 e_2}=0 & (6.27a)\\[4pt]\bar g_{ij}\bar g_{kl}S^i{}_{\underline{c}_2 d_1\underline{d}_2}S^j{}_{\underline{b}_2}{}^k{}_{b_1}S^l{}_{\underline{f}_2 e_1 e_2}=0 & (6.27b)\\[4pt]\bar g_{ij}\bar g_{kl}S^i{}_{\underline{c}_2 d_1\underline{d}_2}S^j{}_{e_1}{}^k{}_{e_2}S^l{}_{\underline{f}_2 b_1\underline{b}_2}=0\,. & (6.27c)\end{cases}$$

(iv) *The corresponding algebraic curvature tensor R satisfies*

$$\boxed{\begin{smallmatrix}b_2\\\hline c_2\\\hline d_2\\\hline f_2\end{smallmatrix}}\;\boxed{b_1|d_1|e_1|e_2}\,\bar g_{ij}\bar g_{kl}R^i{}_{d_1\underline{c}_2\underline{d}_2}R^j{}_{e_1}{}^k{}_{e_2}R^l{}_{b_1\underline{f}_2\underline{b}_2}=0\,. \tag{6.28}$$

Remark 6.14. *To facilitate the reading of this and subsequent proofs, note that
the names of symmetrised indices are completely irrelevant.*

Proof. (i) Contract (6.24) with $\bar{g}_{kl}S^l{}_{f_2e_1e_2}$, antisymmetrise in c_2, d_2, f_2 and symmetrise in b_2, b_1, d_1, e_1, e_2. This yields

$$\young(c_2,d_2,f_2)\,\young(b_2b_1d_1e_1e_2)\,\bar{g}_{ij}\bar{g}_{kl}$$
$$\left(S^i{}_{b_1}{}^k{}_{b_2}S^j{}_{\underline{c_2}d_1\underline{d_2}} - \tfrac{1}{2}S^i{}_{d_1}{}^k{}_{\underline{d_2}}S^j{}_{\underline{c_2}b_1b_2} - S^i{}_{\underline{d_2}}{}^k{}_{d_1}S^j{}_{\underline{c_2}b_1b_2}\right)S^l{}_{\underline{f_2}e_1e_2} = 0\,.$$

Reordering indices shows that the third term differs from the second by a factor of minus two. Indeed, exchanging d_1 and d_2 in the third term is tantamount to exchanging the upper indices i and k, due to the pair symmetry of $S^i{}_{d_2}{}^k{}_{d_1}$. But under contraction with $\bar{g}_{ij}\bar{g}_{kl}$ this is tantamount to exchanging the upper indices j and l. This in turn is tantamount to exchanging c_2, b_1, b_2 with f_2, e_1, e_2 which, under symmetrisation and antisymmetrisation, is tantamount to a sign change. Therefore

$$\young(c_2,d_2,f_2)\,\young(b_2b_1d_1e_1e_2)\,\bar{g}_{ij}\bar{g}_{kl}\left(S^i{}_{b_1}{}^k{}_{b_2}S^j{}_{\underline{c_2}d_1\underline{d_2}} + \tfrac{1}{2}S^i{}_{d_1}{}^k{}_{\underline{d_2}}S^j{}_{\underline{c_2}b_1b_2}\right)S^l{}_{\underline{f_2}e_1e_2} = 0\,. \qquad (6.29)$$

Applying the symmetrised Bianchi identity to $S^i{}_{b_1}{}^k{}_{b_2}$ and antisymmetrising in b_2, c_2, d_2, f_2 yields

$$\young(b_2b_1d_1e_1e_2,c_2,d_2,f_2)^{\star}\,\bar{g}_{ij}\bar{g}_{kl}\left(S^{ik}{}_{b_1b_2}S^j{}_{c_2d_1d_2} - S^i{}_{d_1}{}^k{}_{d_2}S^j{}_{c_2b_1b_2}\right)S^l{}_{f_2e_1e_2} = 0\,.$$

We have derived this identity from the first integrability condition via lemma 6.12. Comparing it with condition (6.20) shows that (6.20) is equivalent to (6.25b) and, after using once again the symmetrised Bianchi identity, also to (6.25a). This proves (i), since we have already shown that the second integrability condition is equivalent to (6.20).

(ii) Condition (6.26a) is equivalent to (6.25a). This results from (6.5) when taking (6.18a) and (6.15) into account. In the same way (6.26b) is equivalent to (6.25b), using (6.18b).

(iii) We will prove the equivalence of (6.25a) to each of the equations (6.27). To this aim we establish three linearly independent homogeneous equations for the three tensors on the left hand side of (6.27). For the first equation we split the hook symmetriser in (6.25a) at the label b_2 and get

$$\young(b_2,c_2,d_2,f_2)\,\young(b_1d_1e_1e_2)\,\bar{g}_{ij}\bar{g}_{kl}\Big(\ S^i{}_{\underline{c_2}d_1\underline{d_2}}S^j{}_{b_1}{}^k{}_{\underline{b_2}}S^l{}_{\underline{f_2}e_1e_2}$$
$$+S^i{}_{\underline{c_2}e_1\underline{d_2}}S^j{}_{d_1}{}^k{}_{b_1}S^l{}_{\underline{f_2}e_2\underline{b_2}}$$
$$+S^i{}_{\underline{c_2}e_2\underline{d_2}}S^j{}_{e_1}{}^k{}_{d_1}S^l{}_{\underline{f_2}\underline{b_2}b_1}$$
$$+S^i{}_{\underline{c_2}\underline{b_2}\underline{d_2}}S^j{}_{e_2}{}^k{}_{e_1}S^l{}_{\underline{f_2}b_1d_1}$$
$$+S^i{}_{\underline{c_2}b_1\underline{d_2}}S^j{}_{\underline{b_2}}{}^k{}_{e_2}S^l{}_{\underline{f_2}d_1e_1}\Big) = 0\,.$$

The fourth term vanishes by the Bianchi identity and the second term is equal to the third. Therefore (6.25a) is equivalent to

$$\young(b_2,c_2,d_2,f_2)\,\young(b_1d_1e_1e_2)\,\bar{g}_{ij}\bar{g}_{kl}S^i{}_{\underline{c}_2 d_1 \underline{d}_2} \tag{6.30a}$$
$$\left(S^j{}_{b_1}{}^k{}_{\underline{b}_2} S^l{}_{\underline{f}_2 e_1 e_2} + 2S^j{}_{e_1}{}^k{}_{e_2} S^l{}_{\underline{f}_2 b_1 \underline{b}_2} + S^j{}_{\underline{b}_2}{}^k{}_{b_1} S^l{}_{\underline{f}_2 e_1 e_2}\right) = 0\,.$$

This is our first equation. The other two equations follow from the first integrability condition as follows. The second equation is obtained from (6.22) by contracting with $\bar{g}_{kl}S^l{}_{f_2 e_1 e_2}$, antisymmetrising in b_2, c_2, d_2, f_2 and symmetrising in b_1, d_1, e_1, e_2:

$$\young(b_2,c_2,d_2,f_2)\,\young(b_1d_1e_1e_2)\,\bar{g}_{ij}\bar{g}_{kl}\left(S^i{}_{b_1}{}^k{}_{\underline{b}_2} + 2S^i{}_{\underline{b}_2}{}^k{}_{b_1}\right)S^j{}_{\underline{c}_2 d_1 \underline{d}_2}S^l{}_{\underline{f}_2 e_1 e_2} = 0\,.$$

This can be rewritten as

$$\young(b_2,c_2,d_2,f_2)\,\young(b_1d_1e_1e_2)\,\bar{g}_{ij}\bar{g}_{kl}S^i{}_{\underline{c}_2 d_1 \underline{d}_2}\left(S^j{}_{b_1}{}^k{}_{\underline{b}_2} S^l{}_{\underline{f}_2 e_1 e_2} + 2S^j{}_{\underline{b}_2}{}^k{}_{b_1} S^l{}_{\underline{f}_2 e_1 e_2}\right) = 0 \tag{6.30b}$$

and is our second equation. For the third equation, we rename b_1, b_2 in (6.24) as e_1, e_2, contract with $\bar{g}_{kl}S^l{}_{f_2 b_1 b_2}$, antisymmetrise in b_2, c_2, d_2, f_2 and symmetrise in b_1, d_1, e_1, e_2:

$$\young(b_2,c_2,d_2,f_2)\,\young(b_1d_1e_1e_2)\,\bar{g}_{ij}\bar{g}_{kl}S^l{}_{\underline{f}_2 b_1 \underline{b}_2}$$
$$\left(S^i{}_{e_1}{}^k{}_{e_2} S^j{}_{\underline{c}_2 d_1 \underline{d}_2} - \tfrac{1}{2}S^i{}_{d_1}{}^k{}_{\underline{d}_2} S^j{}_{\underline{c}_2 e_1 e_2} - S^i{}_{\underline{d}_2}{}^k{}_{d_1} S^j{}_{\underline{c}_2 e_1 e_2}\right) = 0\,.$$

This can be rewritten as

$$\young(b_2,c_2,d_2,f_2)\,\young(b_1d_1e_1e_2)\,\bar{g}_{ij}\bar{g}_{kl}S^i{}_{\underline{c}_2 d_1 \underline{d}_2} \tag{6.30c}$$
$$\left(S^j{}_{e_1}{}^k{}_{e_2} S^l{}_{\underline{f}_2 b_1 \underline{b}_2} - \tfrac{1}{2}S^j{}_{\underline{b}_2}{}^k{}_{b_1} S^l{}_{\underline{f}_2 e_1 e_2} - S^j{}_{b_1}{}^k{}_{\underline{b}_2} S^l{}_{\underline{f}_2 e_1 e_2}\right) = 0$$

and is our last equation. Clearly, the resulting homogeneous system (6.30) implies (6.27). On the other hand, any of the equations (6.27) together with (6.30b) and (6.30c) implies (6.30a) and therefore (6.25a).

(iv) Condition (6.28) is equivalent to (6.27c) via (4.14). $\qquad\square$

6.3. Redundancy of the third integrability condition.
The aim of this section is to prove the following:

Proposition 6.15 (Third integrability condition). *For a Killing tensor on a constant sectional curvature manifold the third of the three integrability conditions (2.3) is redundant.*

We have already shown that the third integrability condition is equivalent to (6.21). As before we can infer from (6.5) together with (6.19) and (6.15) that (6.21) is equivalent to

$$\young(c_2,d_2,f_2)\,\young(b_2b_1d_1e_1e_2g_1g_2)\,\bar{g}_{ij}\bar{g}_{kl}\bar{g}_{mn} \tag{6.31}$$
$$\left(S^{ik}{}_{b_1 b_2} S^j{}_{\underline{c}_2 d_1 \underline{d}_2} + S^i{}_{\underline{c}_2 b_1 b_2} S^j{}_{d_1}{}^k{}_{\underline{d}_2}\right)S^m{}_{\underline{f}_2 e_1 e_2} S^{nl}{}_{g_1 g_2} = 0\,.$$

The proceeding to prove this equation is similar to the proof of part (iii) in proposition 6.13. From the first two integrability conditions we will deduce the following three equations

$$(6.33\text{a})-\tfrac{1}{2}(6.33\text{b})-(6.33\text{c})=0 \tag{6.32a}$$

$$2(6.33\text{b})+(6.33\text{c})=0 \tag{6.32b}$$

$$(6.33\text{b})-(6.33\text{c})=0 \tag{6.32c}$$

for the tensors

$$
\begin{cases}
\bar{g}_{ij}\bar{g}_{kl}\bar{g}_{mn}S^i{}_{\underline{c_2}d_1\underline{d_2}}\,S^j{}_{b_1}{}^k{}_{b_2}\,S^l{}_{g_1}{}^m{}_{g_2}\,S^n{}_{\underline{f_2}e_1e_2} & (6.33\text{a}) \\[2mm]
\bar{g}_{ij}\bar{g}_{kl}\bar{g}_{mn}S^i{}_{\underline{c_2}b_1b_2}\,S^j{}_{d_1}{}^k{}_{\underline{d_2}}\,S^l{}_{g_1}{}^m{}_{g_2}\,S^n{}_{\underline{f_2}e_1e_2} & (6.33\text{b}) \\[2mm]
\bar{g}_{ij}\bar{g}_{kl}\bar{g}_{mn}S^i{}_{\underline{c_2}b_1b_2}\,S^j{}_{\underline{d_2}}{}^k{}_{d_1}\,S^l{}_{g_1}{}^m{}_{g_2}\,S^n{}_{\underline{f_2}e_1e_2}\,. & (6.33\text{c})
\end{cases}
$$

The system (6.32) shows that each of the tensors (6.33) is zero. In particular this proves our claim, since (6.31) can be written as a linear combination of these tensors.

6.3.1. *First equation.* Contract (6.24) with $\bar{g}_{kl}\bar{g}_{mn}S^l{}_{g_1}{}^m{}_{g_2}S^n{}_{f_2e_1e_2}$, antisymmetrise in c_2, d_2, f_2 and symmetrise in the remaining seven indices:

$$\bar{g}_{ij}\bar{g}_{kl}\bar{g}_{mn}$$
$$\left(S^i{}_{b_1}{}^k{}_{b_2}S^j{}_{\underline{c_2}d_1\underline{d_2}} - \tfrac{1}{2}S^i{}_{d_1}{}^k{}_{\underline{d_2}}S^j{}_{\underline{c_2}b_1b_2} - S^i{}_{\underline{d_2}}{}^k{}_{d_1}S^j{}_{\underline{c_2}b_1b_2}\right)S^l{}_{g_1}{}^m{}_{g_2}S^n{}_{\underline{f_2}e_1e_2} = 0\,.$$

Renaming i, j as j, i, this can be written as

$$\bar{g}_{ij}\bar{g}_{kl}\bar{g}_{mn}$$
$$\left(S^i{}_{\underline{c_2}d_1\underline{d_2}}S^j{}_{b_1}{}^k{}_{b_2} - \tfrac{1}{2}S^i{}_{\underline{c_2}b_1b_2}S^j{}_{d_1}{}^k{}_{\underline{d_2}} - S^i{}_{\underline{c_2}b_1b_2}S^j{}_{\underline{d_2}}{}^k{}_{d_1}\right)S^l{}_{g_1}{}^m{}_{g_2}S^n{}_{\underline{f_2}e_1e_2} = 0\,.$$

This is our first equation (6.32a).

6.3.2. *Second equation.* Reduce the antisymmetriser in (6.24) completely,

$$\bar{g}_{ij}\left(S^i{}_{b_1}{}^k{}_{b_2}S^j{}_{c_2d_1d_2} - \tfrac{1}{2}S^i{}_{d_1}{}^k{}_{d_2}S^j{}_{c_2b_1b_2} - S^i{}_{d_2}{}^k{}_{d_1}S^j{}_{c_2b_1b_2}\right.$$
$$\left.-S^i{}_{b_1}{}^k{}_{b_2}S^j{}_{d_2d_1c_2} + \tfrac{1}{2}S^i{}_{d_1}{}^k{}_{c_2}S^j{}_{d_2b_1b_2} + S^i{}_{c_2}{}^k{}_{d_1}S^j{}_{d_2b_1b_2}\right) = 0\,,$$

raise the index c_2, rename it as m and bring the last two terms to the right hand side,

$$\bar{g}_{ij}\left(S^i{}_{b_1}{}^k{}_{b_2}S^{jm}{}_{d_1d_2} - \tfrac{1}{2}S^i{}_{d_1}{}^k{}_{d_2}S^{jm}{}_{b_1b_2} - S^i{}_{d_2}{}^k{}_{d_1}S^{jm}{}_{b_1b_2} - S^i{}_{b_1}{}^k{}_{b_2}S^j{}_{d_2d_1}{}^m\right)$$
$$= -\bar{g}_{ij}\left(\tfrac{1}{2}S^i{}_{d_1}{}^{km}S^j{}_{d_2b_1b_2} + S^{imk}{}_{d_1}S^j{}_{d_2b_1b_2}\right)\,,$$

contract with $\bar{g}_{kl}\bar{g}_{mn}S^l{}_{f_2e_1e_2}S^n{}_{h_2g_1g_2}$, antisymmetrise in d_2, f_2, h_2 and symmetrise in the remaining seven indices:

$$\bar{g}_{ij}\bar{g}_{kl}\bar{g}_{mn}S^l{}_{\underline{f_2}e_1e_2}S^n{}_{\underline{h_2}g_1g_2}$$
$$\left(S^i{}_{b_1}{}^k{}_{b_2}S^{jm}{}_{d_1\underline{d_2}} - \tfrac{1}{2}S^i{}_{d_1}{}^k{}_{\underline{d_2}}S^{jm}{}_{b_1b_2} - S^i{}_{\underline{d_2}}{}^k{}_{d_1}S^{jm}{}_{b_1b_2} - S^i{}_{b_1}{}^k{}_{b_2}S^j{}_{\underline{d_2}d_1}{}^m\right)$$
$$= -\bar{g}_{ij}\bar{g}_{kl}\bar{g}_{mn}$$
$$\left(\tfrac{1}{2}S^i{}_{d_1}{}^{km}S^j{}_{\underline{d_2}b_1b_2} + S^{imk}{}_{d_1}S^j{}_{\underline{d_2}b_1b_2}\right)S^l{}_{\underline{f_2}e_1e_2}S^n{}_{\underline{h_2}g_1g_2}\,.$$

On the right hand side the upper indices j, n, l are implicitly antisymmetrised by the symmetrisation in $b_1, b_2, e_1, e_2, g_1, g_2$ and the antisymmetrisation in d_2, f_2, h_2. Due to the term $\bar{g}_{ij}\bar{g}_{kl}\bar{g}_{mn}$ the same holds for the upper indices i, m, k. The Bianchi identity therefore implies that the right hand side is zero. On the left hand side, the Bianchi identity allows us to bring the index m in each term to the third position:

$$\young(d_2,f_2,h_2)\;\young(b_2b_1d_1e_1e_2g_1g_2)\;\bar{g}_{ij}\bar{g}_{kl}\bar{g}_{mn}$$
$$S^l{}_{\underline{f}_2 e_1 e_2}\left(-S^i{}_{b_1}{}^k{}_{b_2}S^j{}_{d_1}{}^m{}_{\underline{d}_2} + S^i{}_{d_1}{}^k{}_{\underline{d}_2}S^j{}_{b_1}{}^m{}_{b_2}\right.$$
$$\left. -2S^i{}_{b_1}{}^k{}_{b_2}S^j{}_{\underline{d}_2}{}^m{}_{d_1} +2S^i{}_{\underline{d}_2}{}^k{}_{d_1}S^j{}_{b_1}{}^m{}_{b_2}\right)S^n{}_{\underline{h}_2 g_1 g_2} = 0\,.$$

Using pair symmetry and renaming the indices i, j, k, l as k, l, j, i, this can be written as

$$\young(d_2,f_2,h_2)\;\young(b_2b_1d_1e_1e_2g_1g_2)\;\bar{g}_{ij}\bar{g}_{kl}\bar{g}_{mn}$$
$$S^i{}_{\underline{f}_2 e_1 e_2}\left(-S^j{}_{b_2}{}^k{}_{b_1}S^l{}_{d_1}{}^m{}_{\underline{d}_2} + S^j{}_{\underline{d}_2}{}^k{}_{d_1}S^l{}_{b_1}{}^m{}_{b_2}\right.$$
$$\left. -2S^j{}_{b_2}{}^k{}_{b_1}S^l{}_{\underline{d}_2}{}^m{}_{d_1} +2S^j{}_{d_1}{}^k{}_{\underline{d}_2}S^l{}_{b_1}{}^m{}_{b_2}\right)S^n{}_{\underline{h}_2 g_1 g_2} = 0$$

Renaming i, j, k, l, m, n in reverse order, the first term can be seen to differ from the second by a sign. The same is true for the third and fourth term, resulting in

$$\young(d_2,f_2,h_2)\;\young(b_2b_1d_1e_1e_2g_1g_2)\;\bar{g}_{ij}\bar{g}_{kl}\bar{g}_{mn}$$
$$S^i{}_{\underline{f}_2 e_1 e_2}\left(S^j{}_{\underline{d}_2}{}^k{}_{d_1}S^l{}_{b_1}{}^m{}_{b_2} + 2S^j{}_{d_1}{}^k{}_{\underline{d}_2}S^l{}_{b_1}{}^m{}_{b_2}\right)S^n{}_{\underline{h}_2 g_1 g_2} = 0\,.$$

This is our second equation (6.32b).

6.3.3. *Third equation.* Take the second integrability condition in the form (6.26b), reduce the antisymmetriser by the label f_2,

$$\young(c_2,d_2)\;\young(b_2b_1d_1e_1e_2)\;\bar{g}_{ij}\bar{g}_{kl}\left(S^i{}_{\underline{c}_2 b_1 b_2}S^j{}_{d_1}{}^k{}_{\underline{d}_2}S^l{}_{f_2 e_1 e_2}\right.$$
$$+S^i{}_{\underline{d}_2 b_1 b_2}S^j{}_{d_1}{}^k{}_{f_2}S^l{}_{\underline{c}_2 e_1 e_2}$$
$$\left. +S^i{}_{f_2 b_1 b_2}S^j{}_{d_1}{}^k{}_{\underline{c}_2}S^l{}_{\underline{d}_2 e_1 e_2}\right) = 0\,,$$

raise the index f_2, rename it as m and bring the second term to the right hand side,

$$\young(c_2,d_2)\;\young(b_2b_1d_1e_1e_2)\;\bar{g}_{ij}\bar{g}_{kl}\left(S^i{}_{\underline{c}_2 b_1 b_2}S^j{}_{d_1}{}^k{}_{\underline{d}_2}S^{lm}{}_{e_1 e_2} + S^{im}{}_{b_1 b_2}S^j{}_{d_1}{}^k{}_{\underline{c}_2}S^l{}_{\underline{d}_2 e_1 e_2}\right)$$
$$= -\young(c_2,d_2)\;\young(b_2b_1d_1e_1e_2)\;\bar{g}_{ij}\bar{g}_{kl}S^i{}_{\underline{d}_2 b_1 b_2}S^j{}_{d_1}{}^{km}S^l{}_{\underline{c}_2 e_1 e_2}\,,$$

contract with $\bar{g}_{mn}S^n{}_{f_2 g_1 g_2}$, antisymmetrise in c_2, d_2, f_2 and symmetrise in the remaining seven indices:

$$\young(c_2,d_2,f_2)\;\young(b_2b_1d_1e_1e_2g_1g_2)\;\bar{g}_{ij}\bar{g}_{kl}\bar{g}_{mn}$$
$$\left(S^i{}_{\underline{c}_2 b_1 b_2}S^j{}_{d_1}{}^k{}_{\underline{d}_2}S^{lm}{}_{e_1 e_2} + S^{im}{}_{b_1 b_2}S^j{}_{d_1}{}^k{}_{\underline{c}_2}S^l{}_{\underline{d}_2 e_1 e_2}\right)S^n{}_{\underline{f}_2 g_1 g_2}$$
$$= -\young(c_2,d_2,f_2)\;\young(b_2b_1d_1e_1e_2g_1g_2)\;\bar{g}_{ij}\bar{g}_{kl}\bar{g}_{mn}S^i{}_{\underline{d}_2 b_1 b_2}S^j{}_{d_1}{}^{km}S^l{}_{\underline{c}_2 e_1 e_2}S^n{}_{\underline{f}_2 g_1 g_2}\,.$$

As before, the right hand side is zero. On the left hand side use the symmetrised Bianchi identity to move the upper index m to the third position:

$$\young(c_2,d_2,f_2)\;\young(b_2b_1d_1e_1e_2g_1g_2)\;\bar{g}_{ij}\bar{g}_{kl}\bar{g}_{mn}$$

$$\bigl(S^i{}_{\underline{c_2}b_1b_2}S^j{}_{d_1}{}^k{}_{\underline{d_2}}S^l{}_{e_1}{}^m{}_{e_2} + S^i{}_{b_1}{}^m{}_{b_2}S^j{}_{d_1}{}^k{}_{\underline{c_2}}S^l{}_{\underline{d_2}e_1e_2}\bigr)S^n{}_{\underline{f_2}g_1g_2} = 0\,.$$

Using pair symmetry and renaming the upper indices in the second term, this can be written as

$$\young(c_2,d_2,f_2)\;\young(b_2b_1d_1e_1e_2g_1g_2)\;\bar{g}_{ij}\bar{g}_{kl}\bar{g}_{mn}$$

$$\bigl(S^i{}_{\underline{c_2}b_1b_2}S^j{}_{d_1}{}^k{}_{\underline{d_2}}S^l{}_{e_1}{}^m{}_{e_2} + S^i{}_{\underline{d_2}e_1e_2}S^j{}_{\underline{c_2}}{}^k{}_{d_1}S^l{}_{b_1}{}^m{}_{b_2}\bigr)S^n{}_{\underline{f_2}g_1g_2} = 0\,.$$

This is our third and last equation (6.32c).

7. Application

Finally, we show that the family (1.2) satisfies the algebraic integrability conditions (1.1) and therefore describes integrable Killing tensors.

Proof of the Main Corollary. We will write a dot in place of each index whose name is irrelevant for our considerations. Written in components, the algebraic curvature tensor

$$R = \lambda_2 h \owedge h + \lambda_1 h \owedge g + \lambda_0 g \owedge g$$

is then a linear combination of tensors of the form $h_{..}h_{..}$, $h_{..}g_{..}$ and $g_{..}g_{..}$ or, written in another way, of the form $A_{..}B_{..}$ with $A,B \in \{g,h\}$. Then $R_{a_1b_1a_2b_2}R_{c_1d_1c_2d_2}$ is a linear combination of terms of the form $A_{..}B_{..}C_{..}D_{..}$ with $A,B,C,D \in \{g,h\}$ and thus $g_{ij}R^i{}_{b_1a_2b_2}R^i{}_{d_1c_2d_2}$ is a linear combination of terms of the form $A_{..}B_{..}C_{..}$ with $A,B,C \in \{g,h,h^2\}$. Here g, h and h^2 are symmetric tensors, where h^2 denotes the tensor $g_{ij}h^i{}_ah^j{}_b$. Therefore the antisymmetrisation of $A_{..}B_{..}C_{..}$ in four of the six indices vanishes by Dirichlet's drawer principle. This proves that the first integrability condition (1.1a) is satisfied.

In the same way, the tensor $R_{....}R_{....}R_{....}$ is a linear combination of terms of the form $A_{..}B_{..}C_{..}D_{..}E_{..}F_{..}$ with $A,\dots,F \in \{g,h\}$. Hence the tensor

$$g_{ij}g_{kl}R^i{}_{b_1a_2b_2}R^j{}_{a_1}{}^k{}_{c_1}R^l{}_{d_1c_2d_2}$$

is a linear combination of terms of the form $A_{..}B_{..}C_{..}D_{..}$ with either $A,B,C,D \in \{g,h,h^2\}$ or $A,B,C,D \in \{g,h,h^3\}$, where h^3 denotes the symmetric tensor

$$g_{ij}g_{kl}h^i{}_ah^{jk}h^l{}_b\,.$$

Without loss of generality we may suppose $D = C$, owing to the drawer principle. Consider therefore the tensor $A_{..}B_{..}C_{..}C_{..}$ under antisymmetrisation in four of its indices and symmetrisation in the remaining four. The result vanishes trivially if the antisymmetrisation includes an index pair of one of the symmetric tensors A,B,C. Otherwise it can be written as

$$\young(a_2,b_2,c_2,d_2)\;\young(a_1b_1c_1d_1)\;A_{a_1a_2}B_{b_1b_2}C_{c_1c_2}C_{d_1d_2}$$

and vanishes too, which becomes evident when c_1, c_2 is renamed as d_1, d_2. This demonstrates that the second algebraic integrability condition (1.1b) is also satisfied. □

Remark 7.1. *The family* (1.2) *properly extends Benenti tensors, given by*

$$\lambda_0 = \lambda_1 = 0 \qquad \lambda_2 = \tfrac{1}{2} \qquad h = Ag \qquad A \in \mathrm{GL}(V).$$

Indeed, a Killing tensor corresponding to $h \oslash g$ with $\operatorname{tr} h < 0$ for example is not a Benenti tensor, as can be seen by comparing the scalar curvatures

$$\mathrm{Scal}\big(\tfrac{1}{2}(Ag) \oslash (Ag)\big) = \operatorname{tr}^2(Ag) - \operatorname{tr}(Ag)^2 = \operatorname{tr}^2(A^TA) - \operatorname{tr}(A^TAA^TA)$$
$$= \|A\|^4 - \|A^TA\|^2 \geqslant 0$$

and

$$\mathrm{Scal}(h \oslash g) = 2(N-1)\operatorname{tr} h < 0.$$

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